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A Lyapunov Function for the Dynamical Study of Axial Flow Compressor Dynamics

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***Abstract*—This paper constructs a class of quadratic-type Lyapunov functions for the study of axial flow compressor dynamics. Based on Moore-Greitzer's model proposed in 1986, conditions for local system stability is carried out via the proposed Lyapunov function for the compression system without considering the dynamics of stall wave. The domain of attraction for compression system is also analytically estimated. A sufficient condition is also obtained to guarantee the global stability of some un-stalled normal operating equilibria. Case study for both compressors with cubic axisymmetric characteristics and axisymmetric characteristic proposed by Liaw and Abed in 1991 are discussed to verify the usage of the analytical results. Numerical simulations for both cases are also obtained to demonstrate the main results.**

***Index Terms*—Axial flow compressor, Lyapunov function, stability.**

I. INTRODUCTION

In the recent years, the study of axial flow compressor dynamics has attracted lots of attention mainly due to the requirement of high system performance. It is known that two types of unstable phenomena might occur while the compressor operates near its maximum achievable pressure rise. Those nonlinear behaviors might limit the performance and/or operation of gas turbine jet engines. Under either of the two unstable conditions, a moderate disturbance can result in system instability so that the compression system might experience a large amplitude oscillation (corresponding to the so-called "surge") or jump into a very inefficient operation at constant mass flux with low pressure-rise (corresponding to the so-called "stall") (e.g., [1]-[4]). Among those previous studies, in 1976 Greitzer proposed a four-dimensional lumped parameter model to describe the behavior of surge and stall behavior [1]. A third-order differential equation model is then proposed by Moore and Greitzer in 1986 to capture the major characteristics of both surge and rotating stall phenomena occurring in axial flow compression systems [2].

Based on Moore and Greitzer's model [2], the stall behavior of axial flow compression system was found to be attributed to the occurrence of the so-called "transcritical stationary bifurcation" [3] or the so-called "pitch-fork stationary bifurcation" [5] depending on the representation of system state. In practical applications, a surge (or stall) line is usually set up to provide a safe operational boundary for the usage of compressors. Such a conservative trade-off unduly restricts the capability of compressor and/or engine. Various types of control schemes have been proposed to allow compressors to operate safely beyond the surge line and thus enhance system efficiency (e.g., [6]-[13]). Among those designs, several robust control schemes had been applied to deal with system uncertainties by assuming the axisymmetric characteristic to be a specific cubic function (e.g., [6], [11]-[13].

It is known that the determination of stability regions by using Lyapunov analysis plays an important role such as in stabilization and robustness analysis (e.g., [12]-[15]). Among those studies, Simon [14] constructed an energy function for the one-dimensional model of the axial compressor dynamics to develop a nonlinear robust control law. The domain of attraction was also loosely discussed. Mansoux et. al. [15] proposed a quadratic type Lyapunov function for the analysis of two-dimensional model of the compressor dynamics. In particular, the domain of attraction was semi-empirically determined. In our previous study [5], we had presented the study of the three-dimensional stall behaviour in axial flow compressors by using Moore-Greitzer's model proposed in [2]. However, no results regarding the domain of attraction for the corresponding equilibrium has been discussed in the current studies. In this paper, we will extend previous results to focus on the study of the two-dimensional behavior in compressor dynamics by using the same model without considering the dynamics of stall wave. A preliminary result of this study was presented in [16]-[17] by using extensive computer simulations with the help of the code AUTO [18] for the axial flow compressor dynamics with respect to the variations of system parameters γ and B. Instead of using numerical approach in [16]-[17], in this study we propose another approach by the construction of Lyapunov functions for determining the local stability of system equilibrium and the corresponding domain of attraction.

This paper is organized as follows. In Section II, the mathematical model of compression systems introduced by Moore and Greitzer in 1986 is recalled. It is followed by the construction of a class of quadratic Lyapunov function for the 1986 year's model without considering the dynamics of stall wave. Both of local stability criteria and the domain of attraction are analyzed and estimated for axial compressors. Numerical simulations will be given in Section IV for compressors with cubic axisymmetric characteristics and axisymmetric characteristic proposed by Liaw and Abed in [16] to verify the usage of the analytical results. Finally,

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Section V gives the main conclusions.

Notation:

A amplitude of the first angular mode of rotating stallwave;
$\dot{m}_C$ nondimensional compressor mass flow rate;
ΔP nondimensional plenum pressure rise;
θ angle along circumference;
α a geometry-related constant;
W semi-width of cubic characteristic
C_{ss} nondimensional axisymmetric compressor characteristic;
F inverse function of nondimensional throttle pressure rise;
B Greitzer B parameter, proportional to rotor speed and plenum volume;
γ control parameter of throttle function;

II. Axial Flow Compressor Dynamics

In this section, we will recall the mathematical model of axial flow compressor dynamics proposed by Moore and Greitzer in 1986 [2]. A previous result regarding the study of rotating stall [5] is also briefly reviewed, which will then be used in the sequel.

Conceptually, a compression system majorly comprises inlet duct, compressor, exit duct, plenum, and throttle as depicted in Fig. 1. In 1986, Moore and Greitzer [2] extended their previous study of [1] to propose a third-order differential equation model for describing the nonlinear behavior of surge and rotating stall phenomena appearing in axial flow compressor dynamics. By adopting the notations of [5], the proposed model in [2] can then be represented as

$$\frac{dA}{dt}=\frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C+WA\sin\theta)\sin\theta\, d\theta, \qquad (1)$$

$$\frac{d\dot{m}_C}{dt}=-\Delta P+\frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C+WA\sin\theta)\, d\theta, \qquad (2)$$

$$\frac{d\Delta P}{dt}=\frac{1}{4B^2}\{\dot{m}_C-F(\gamma,\Delta P)\}, \qquad (3)$$

where the system variables in Eqs. (1)-(3) can be referred to the notations defined above. It is known (e.g., [3]) that the inverse function of nondimensional throttle pressure rise is usually taken as

$$F(\gamma,\Delta P)=\gamma(\Delta P)^{1/2}. \qquad (4)$$

Besides, the axisymmetric compressor characteristic $C_{ss}(\cdot)$ characterizes the steady pressure rise across the compressor and is often known to be an S-shaped function (e.g., [2]-[3]). An example is depicted in Fig. 2.

Suppose $C_{ss}(\cdot)$ is a smooth function. The equilibrium points of system (1)-(3) can then be solved as follows. It is observed from Eq. (1) that $A=0$ results in ${dA}/{dt}=0$. Let $x^0=(0,\dot{m}_C^0,\Delta P^0)$ denote an equilibrium point for the system (1)-(3). By letting the time derivatives in (2)-(3) be zero, we then have $\Delta P^0=C_{ss}(\dot{m}_C^0)$ and $\dot{m}_C^0=F(\gamma^0,\Delta P^0)$ for $A=0$ with a given $\gamma=\gamma^0$. That means system equilibrium of the un-stalled operation (i.e., $A=0$) will be at the intersection point of the throttle line and the compressor characteristic. Note that, the compression system (1)-(3) may have stalled equilibrium (i.e., $A\neq 0$). To adopt the results from [5] as shown in Fig. 2, the throttle line intersects the compressor characteristic at a unique point when the throttle is widely open (i.e., γ is large). In addition, the equilibrium mass flow is found to be decreased while the value of γ is getting smaller. As shown in Fig. 2, solid line stands for stable system equilibria while dot-line stands for unstable ones. Both of local stability criteria and conditions for the existence of rotating stall (i.e., $A\neq 0$) was obtained in [5] as recalled in the next two lemmas.

Lemma 1. Suppose $C_{ss}(\cdot)$ is a smooth function and F is a strictly increasing function in each of its two variables. The equilibrium point x^0 of system (1)-(3) will be asymptotically stable (resp. unstable) if $C_{ss}'(\dot{m}_C^0(\gamma))<0$ (resp. $C_{ss}'(\dot{m}_C^0(\gamma))>0$).

Lemma 2. Suppose $C_{ss}(\cdot)$ is a smooth function and F is a strictly increasing function in each of its two variables. Then the system (1)-(3) will exhibit a pitchfork stationary bifurcation at the equilibrium point x^0 with respect to small variation of γ if $C_{ss}'(\dot{m}_C^0(\gamma))=0$ and $C_{ss}''(\dot{m}_C^0(\gamma))\neq 0$.

It is noted that the condition of determining the characteristic of the bifurcated solutions had been obtained in [5]. A quadratic feedback control law was also proposed to avoid the jumping behavior, which is caused by the occurrence of subcritical pitchfork stationary bifurcation, in compressor dynamics.

III. Construction of Lyapunov Functions

It is known that there are two types of unstable phenomena might occur while the compressor operates near its maximum achievable pressure rise. A moderate disturbance can result in system instability so that the axial flow compression system might either experience a large amplitude oscillation corresponding to the so-called "surge" or jump into a very inefficient operation at constant mass flux and low pressure rise corresponding to the so-called "stall." In our previous study [5] as recalled in Section II, we had presented the study of the three-dimensional stall behaviour in axial flow compressors by using Moore-Greitzer's model [2]. However, no results regarding the domain of attraction for the corresponding equilibrium has been discussed. In this paper, we extend those results to focus on the study of the two-dimensional behavior in compressor dynamics by using the same model as presented in Eqs. (1)-(3) without considering the dynamics of stall wave. Instead of numerical approach used in [16]-[17], in the following we consider another scheme by the construction of Lyapunov functions for determining the local stability of system equilibrium and the estimating the corresponding domain of attraction.

It is observed from Eq. (1) that $A=0$ is an invariant manifold for system (1)-(3). Thus, Eqs. (1)-(3) can then be reduced as a two-dimensional system for the study of non-stall behaviour as given below (with $A=0$) :

$$\frac{d\dot{m}_C}{dt}=C_{ss}(\dot{m}_C)-\Delta P, \qquad (5)$$

$$\frac{d\Delta P}{dt}=\frac{1}{4B^2}\{\dot{m}_C-F(\gamma,\Delta P)\}. \qquad (6)$$

Now, we consider the local stability of system (5)-(6).

Denote $x^0 = \left(\dot m_c^0, \Delta P^0\right)^T$ an equilibrium point of system (5)-(6) for given $\gamma = \gamma^0$. It is clear to have $\Delta P^0 = C_{ss}(\dot m_C^0)$ and $\dot m_C^0 = F(\gamma^0, \Delta P^0)$. Let $x = (x_1, x_2)^T$ with $x_1 = \dot m_c - \dot m_c^0$ and $x_2 = \Delta P - \Delta P^0$.

Then we have the deviation model at the equilibrium x^0 as

$$\dot x_1 = C_{ss}(\dot m_c^0 + x_1) - (\Delta P^0 + x_2) \tag{7}$$

$$\dot x_2 = \frac{1}{4B^2}\{(\dot m_c^0 + x_1) - F(\gamma^0, \Delta P^0 + x_2)\}. \tag{8}$$

First, we recall the Lyapunov stability criterion from (e.g., [19]) as discussed below.

Let $x = 0$ be an equilibrium point for the system given by

$$\dot x = f(x), \text{ for } x \in R^n. \tag{9}$$

Then we have the next stability result for system (9) as recalled in the next lemma.

Lemma 3. The equilibrium point $x = 0$ for system (9) is asymptotically stable if there exists a local region $D \subset R^n$ containing $x = 0$ and a continuous differentiable function $V(x)$ such that the two conditions hold: (i) $V(x)$ is a locally positive definite function, i.e., $V(0) = 0$ and $V(x) > 0$, for all $x \in D$ with $x \neq 0$; and (ii) $\dot V(x) = \nabla V(x) \cdot \dot x < 0$, $\forall\, x \in D$ with $x \neq 0$. Moreover, the local region D denotes the so-called "domain of attraction" (DoA).

To facilitate the derivations, we make the following hypotheses.

Hypothesis 1. Suppose $C_{ss}(\dot m_C)$ and $F(\gamma, \Delta P)$ are smooth functions and satisfy the following assumptions: (i) There exists a continuous region $S_1 \subset R$ such that $C'_{ss}(x) < 0$ for all $x \in S_1$; and (ii) For a given γ^0, there exists a continuous region $S_2 \subset R$ such that $F'\left(\gamma^0, y\right) > 0$ for all $y \in S_2$. Here, $F'\left(\gamma^0, y\right)$ denotes the first derivative of function F with respect to y.

Hypothesis 2. Suppose $C_{ss}(\dot m_C)$ and $F(\gamma, \Delta P)$ are smooth functions and satisfy the following assumptions: (i) For a given $\dot m_c^0$, there exists a continuous region $S_3 \subset R$ containing $\dot m_c^0$, such that $\{C_{ss}(\dot m_c^0 + \eta) - C_{ss}(\dot m_c^0)\} \cdot \eta < 0$ for all η with $(\dot m_c^0 + \eta) \in S_3$; and (ii) For given γ^0 and ΔP^0, there exists a continuous region $S_4 \subset R$ containing ΔP^0 such that $\zeta \cdot \{F(\gamma^0, \Delta P^0 + \zeta) - F(\gamma^0, \Delta P^0)\} > 0$ for all ζ with $(\Delta P^0 + \zeta) \in S_4$.

Note that, the notation R in the above two hypotheses denotes the set of real number.

Choose a Lyapunov function candidate as given by

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + 4B^2 x_2^2). \tag{10}$$

It is clear from (e.g., [19]) that $V(x_1, x_2)$ is a positive definite function for all $(x_1, x_2) \in R^2$. We then have the time derivative of the function V as follows:

$$\begin{aligned}\dot V(x_1, x_2) &= x_1 \dot x_1 + 4B^2 x_2 \dot x_2 \\ &= x_1 \cdot \{C_{ss}(x_1 + \dot m_c^0) - C_{ss}(\dot m_c^0)\} \\ &\quad - x_2 \cdot \{F(\gamma^0, \Delta P^0 + x_2) - F(\gamma^0, \Delta P^0)\}.\end{aligned} \tag{11}$$

By using Taylor series expansion, we can have the next stability result.

Lemma 4. The equilibrium point x^0 of system (5)-(6) is asymptotically stable if $C'_{ss}(\dot m_c^0) < 0$ and $F'(\gamma^0, \Delta P^0) < 0$.

Proof:

By Taylor series expansion, we have

$$C_{ss}(x_1 + \dot m_c^0) \cong C_{ss}(\dot m_c^0) + C'_{ss}(\dot m_c^0) x_1 \tag{12}$$

and $F(\gamma^0, \Delta P^0 + x_2) \cong F(\gamma^0, \Delta P^0) + F'(\gamma^0, \Delta P^0) x_2$ (13)

for all $(x_1, x_2) \in \Omega(x^0)$. Here, $\Omega(x^0)$ denotes a local neighborhood of the equilibrium point x^0.

Eq. (11) can then be rewritten as

$$\dot V(x_1, x_2) \cong C_{ss}'(\dot m_c^0) x_1^2 - F'(\gamma^0, \Delta P^0) x_2^2 \tag{14}$$

It is clear that $\dot V(x_1, x_2) < 0$ for all $(x_1, x_2) \in \Omega(x^0)$ with $(x_1, x_2) \neq (0,0)$. The result of the lemma can then be implied by Lyapunov stability criteria recalled in Lemma 3. ■

Remark 1. The result obtained in Lemma 4 agrees with that in [10] by using system eigenvalues approach.

Note that, the domain of attraction (DoA) $\Omega(x^0)$ in general is not easy to obtain from Lemma 4. In the following, we will propose two different approaches for the estimation of DoA. Detains are given below.

Next, we recall the so-called "mean value theorem (MVT)" (e.g., [20]) as given below, which will be used to estimate the region of the DoA.

Lemma 5. Suppose a real valued function $f(x)$ is differentiable on an interval $I @ [x_1,\, x_2] \subset R$ with $x_1 \neq x_2$. Then there exists a point $\eta \in [x_1,\, x_2]$ such that

$$f(x_1) - f(x_2) = f'(\eta) \cdot (x_1 - x_2). \tag{15}$$

Suppose Hypothesis 1 holds. Then from Lemma 5 we have

$$C_{ss}(x_1 + \dot m_c^0) = C_{ss}(\dot m_c^0) + C_{ss}'(\dot m_c^0 + \rho_1 x_1) x_1 \tag{16}$$

and

$$F(\gamma^0, \Delta P^0 + x_2) = F(\gamma^0, \Delta P^0) + F'(\gamma^0, \Delta P^0 + \rho_2 x_2) x_2 \tag{17}$$

for some $\rho_1, \rho_2 \in [0,\, 1]$.

Now, we can rewrite Eq. (11) as

$$\dot V(x_1, x_2) = C'_{ss}(\dot m_c^0 + \rho_1 x_1) x_1^2 - F'(\gamma^0, \Delta P^0 + \rho_2 x_2) x_2^2 < 0 \tag{18}$$

for all $x_1 \in S_1$ and $x_2 \in S_2$.

The result given below is readily implied from Lemma 3.

Theorem 1. Suppose Hypothesis 1 holds. The equilibrium point x^0 of system (5)-(6) is then asymptotically stable with the DoA $\Omega_1(x^0)$ defined by

$$\begin{aligned}\Omega_1(x^0) @ \{(x_1, x_2) \,|\, V(x_1, x_2) \le C_1 \text{ with } (\dot m_c^0 + x_1) \in S_1 \\ \text{and } (\Delta P^0 + x_2) \in S_2\},\end{aligned} \tag{19}$$

where $V(x_1, x_2) = C_1$ denotes the level curve of largest value C_1 on the region $S_1 \cup S_2$ with x^0 as the reference point.

Remark 2. The result given in Lemma 4 can be induced from that of Theorem 1.

Next, we consider the case of which Hypothesis 2 holds. By the assumptions of Hypothesis 2, it is clear from Eq. (11) to have

$$\dot{V}(x_1, x_2) < 0 \tag{20}$$

for all $x_1 \in S_3$ and $x_2 \in S_4$.

We then have the next result from Lemma 3.

Theorem 2. Suppose Hypothesis 2 holds. Then the equilibrium point x^0 of system (5)-(6) is asymptotically stable with the DoA $\Omega_2(x^0)$ defined by

$$\Omega_2(x^0) \triangleq \{(x_1, x_2) \,|\, V(x_1, x_2) \le C_2 \text{ with } (\dot{m}_c^0 + x_1) \in S_3 \text{ and } (\Delta P^0 + x_2) \in S_4\}, \tag{21}$$

where $V(x_1, x_2) = C_2$ denotes the level curve of largest value C_2 on the region $S_3 \cup S_4$ with x^0 as the reference point.

Remark 3. The result given in Lemma 4 can also be induced from that of Theorem 2.

Suppose $C_{ss}(\cdot)$ is a smooth function and F is a strictly increasing function in each of its two variables. We then have the next global stability result from Theorem 2.

Corollary 1. Suppose $C_{ss}(\cdot)$ is a smooth function and F is a strictly increasing function in each of its two variables. The equilibrium point x^0 of system (5)-(6) will be globally asymptotically stable if condition (i) of Hypothesis 2 holds for $S_3 = R$.

IV. Numerical Examples

In the following, we will adopt two axisymmetric compressor characteristics to demonstrate the usage of the results proposed in Theorems 1 and 2. One is the cubic model from [5] as shown in Fig. 2 with the definition given in (22) below, and the other is from [16] as depicted in Fig. 3.

$$C_{ss}(\dot{m}_C) = 1.56 + 1.5(\dot{m}_C - 1) - 0.5(\dot{m}_C - 1)^3. \tag{22}$$

As discussed in Section II, the Moore and Greitzer's model [1] is majorly employed to describe the three-dimensional dynamics of rotating stall. As recalled in Lemma 1, the linear stability of the axisymmetric equilibrium is stable if $C'_{ss}(\dot{m}_C^0(\gamma)) < 0$. The result obtained in Lemma 4 above for the two-dimensional model agrees with that in Lemma 1 recalled from [5]. As stated in Remarks 2 and 3, the local stability conditions given in Lemma 4 can also be induced from either Theorem 1 or Theorem 2. The key difference among those two results is the estimation of the domain of attraction DoA for the given system equilibrium. From nonlinear system point of view, the domain of attraction for stable system equilibrium may vary as the value of system parameters γ and B changes. When the domain of attraction becomes finite, shrinks and disappears, emergences of multi-equilibria and/or limit-cycle type of oscillations may occur as presented in [5]. In the following, the numerical analysis tool Matlab is employed to unveil the domain of attraction DoA for the given system equilibrium with respect to the variation of system parameters γ and B.

In the following, the inverse function of nondimensional throttle pressure rise $F(\gamma, \Delta P)$ is taken from Eq. (4) for numerical study. It is clear to see that the function $F(\gamma, \Delta P)$ is strictly increasing function for all $\Delta P > 0$ and $\gamma > 0$. That means the condition (ii) of Hypothesis 1 or Hypothesis 2 is satisfied with $S_2 = S_4 \triangleq \{x \,|\, x > 0\}$.

Now, we focus on the feasibility study of the two axisymmetric compressor characteristics. First, we consider the case of which the axisymmetric compressor characteristic $C_{ss}(\cdot)$ is defined in Eq. (22). It is observed from Fig. 2 that we have the peak value $\Delta P_M = 2.56$ of $C_{ss}(\cdot)$ at $\dot{m}_c^P = 2.0$ for γ=1.25. Besides, it is also found that $C'_{ss}(x) < 0$ for all $x \in S_1 \triangleq \{x \,|\, x > \dot{m}_c^P\}$. This implies that condition (i) of Hypothesis 1 holds.

For checking the condition (i) of Hypothesis 2, let $x^0 = \left(\dot{m}_c^0,\ \Delta P^0\right)^T$ be an equilibrium point of system (4)-(5). It is observed from Fig. 2 that we have condition (i) of Hypothesis 2 holds for all η with $(\dot{m}_c^0 + \eta) \in S_3$ where

$$S_3 \triangleq \{x \,|\, x > \dot{m}_c^L,\ \text{with } \dot{m}_c^L < \dot{m}_c^0 \text{ and } C_{ss}(\dot{m}_c^L) = \Delta P^0\}. \tag{23}$$

Thus, the results given in Theorems 1 and 2 can then be applied to the compression system (5)-(6) with the cubic model of (22) for the estimation of the DoA. It is known from Lemma 4 and Fig. 2 that the local stability of system equilibrium x^0 can be guaranteed while it lies on the pre-stall region for $1.25 \le \gamma \le 1.463$ or the un-stalled normal region for $\gamma > 1.463$. In the following, we choose four cases for numerical study of which $\gamma^0 = 1.4$, 1.463, 1.6 and 5, respectively, with $B = 0.5$. Details are discussed below.

To facilitate the demonstration of numerical results, in the following simulations we use the green curve and the orange curve to represent the boundary of the estimated DoA $\Omega_1(x^0)$ and $\Omega_2(x^0)$ defined in Theorem 1 and Theorem 2, respectively. In addition, the pink dot-dashed line denotes the time response for the third-order system (3)-(5) with initial $A = 0.5$, while the blue dot-dashed line denotes the time response for the two-dimensional system (7)-(8) with initial $A = 0$, respectively. Besides, $x(0)$ denotes the initial value.

For the case of which $\gamma^0 = 1.4$, we have the equilibrium point $x^0 = (2.21, 2.4899)$ and $\dot{m}_c^L = 1.7752$. Time responses for both of three-dimensional system (1)-(3) and two-dimensional model (5)-(6) with three different initials are shown in Fig. 4. It is observed from Figs. 4(a) and 4(b) that the state trajectories will go to the system equilibrium x^0 as predicted by Theorems 1 and 2 for both of three-dimensional system (1)-(3) and two-dimensional model (5)-(6) with $A = 0.5$ and $A = 0$, respectively. Besides, the time response with the initial outside the estimated DoA might also approach the equilibrium point x^0 as depicted in Fig. 4(c). It is due to the fact that the estimated DoA from given Lyapunov function only contains the sufficient region. Similar scenarios are shown in Figs. 5 and 6 with

$x^0 = (2.28, 2.43)$, $\dot{m}_c^L = 1.69$ for $\gamma^0 = 1.463$ and $x^0 = (2.411, 2.271)$, $\dot{m}_c^L = 1.52$ for $\gamma^0 = 1.6$, respectively. It is interesting to note that we have semi-globally asymptotical stability instead of globally asymptotical stability for the equilibrium point x^0 with large value of γ^0, say $\gamma^0 = 5$. Time responses given in Fig. 7 do demonstrate the result predicted in Corollary 1. The reason for semi-global stability instead of global stability is that condition (i) of Hypothesis 2 holds for all $0 \le \dot{m}_c < \dot{m}_c^0$ and F is a strictly increasing function for $\Delta P > 0$ only.

Next, we consider the case of which the axisymmetric compressor characteristic $C_{ss}(\cdot)$ is defined in Fig. 3. As observed from Fig. 3, we have the peak value $\Delta P_M = 1.252$ of $C_{ss}(\cdot)$ at $\dot{m}_c^P = 0.4648$ for γ=0.4154. In addition, it is found that $C'_{ss}(x) < 0$ for all $x \in S_1 \triangleq \{x \mid x > \dot{m}_c^P\}$ which implies that condition (i) of Hypothesis 1 holds.

Similarly, from Fig. 3 that we have condition (i) of Hypothesis 2 holds for all η with $(\dot{m}_c^0 + \eta) \in S_3$ where S_3 defined in Eq. (23) above. Thus, the results given in Theorems 1 and 2 can also be applied to the compression system (5)-(6) with the compressor model depicted in Fig. 3 for the estimation of the DoA.

In the following, four cases are selected for numerical study of which $\gamma^0 = 0.43$, 0.4421, 0.45 and 1, respectively. In the first three cases we choose $B = 0.4$ and the last one with $B = 0.2$. For simplicity and without loss of generality, in this case we only consider the numerical simulations for the second-order model. Details are discussed below.

Similar scenarios as those for the first case study of cubic characteristic can also be found in Figs. 8-11. Here, we have $x^0 = (0.48, 1.24)$ and $\dot{m}_c^L = 0.457$ for $\gamma^0 = 0.43$, and $\dot{m}_c^L = 0.455$ for $\gamma^0 = 0.442$, $x^0 = (0.4989, 1.2291)$ and $\dot{m}_c^L = 0.4547$ for $\gamma^0 = 0.45$, respectively. In addition, we have semi-globally asymptotical stability for $x^0 = (0.8785, 0.772)$ with $\gamma^0 = 1$ as depicted in Fig. 11. Note that, it is observed in Figs. 8(d) and 8(e) that the timing trajectories might either go to a surge behavior or approach another system equilibrium with different initial conditions. Those two phenomena are attributed to the occurrence of multi-equilibrium in the pre-stall region.

V. Conclusions

In this study, we have proposed a class of Lyapunov function for deriving the local stability of the two-dimensional axial flow compressor dynamics. The result agrees with the one in [10]. In addition, the domain of attraction for the stable system equilibrium is also estimated in terms of the system characteristic. Numerical results have demonstrated the usage of the proposed schemes, which might give a guide for the determination of the so-called ``surge line" in the practical applications.

References

[1] E. M. Greitzer, "Surge and rotating stall in axial flow compressor, Part I: Theoretical compression system model," *ASME J. Engineering for Power*, 1976, pp. 190-198.

[2] F. K. Moore, and E. M. Greitzer, "A theory of post-stall transients in axial compression systems: Part I--Development of equations," *ASME J. Engineering for Gas Turbines and Power*, vol. 108, 1986, pp. 68-76.

[3] F. E. McCaughan, "Application of bifurcation theory to axial flow compressor instability," *ASME J. Turbomachinery*, vol. 111, 1990, pp. 426-433.

[4] E. H. Abed, P. K. Houpt and W. M. Hosny, "Bifurcation analysis of surge and rotating stall in axial flow compressors," *Proc. 1990 American Control Conference*, San Diego, 1990, pp. 2239-2246.

[5] D.-C. Liaw and E. H. Abed, "Active control of compressor stall inception: a bifurcation-theoretic approach," *Autumatica*, vol. 32, 1996, pp. 109-115.

[6] M. Krstic, D. Fontaineand, P. V. Kokotovic, and J. D. Paduano, "Useful nonlinearities and global stabilization of bifurcation in a model of jet engine surge and stall," *IEEE Transactions on Automatic Control*, vol. 43, 1998, pp. 1739-1745.

[7] N. A. Chaturvedi, and S. P. Bhat, "Output-feedback semiglobal stabilization of stall dynamics for preventing hysteresis and surge in axial-flow compressors," *IEEE Transactions on Control Systems Technology*, vol. 14, 2006, pp. 301-307.

[8] D. Fontaine, S. Liao, J. Paduano, and P. V. Kokotovic, "Nonlinear control experiments on an axial flow compressor," *IEEE Transactions on Control Systems Technology*, vol. 12, 2004, pp. 683-693.

[9] M. M. Haddad, J. R. Corrado, and A. Leonessa, "Fixed-order dynamic compensation for axial flow compression systems," *IEEE Transactions on Control Systems Technology*, vol. 10, 2002, pp. 727-734.

[10] D.-C. Liaw, Y.-H. Huang, and W.-C. Chung, "Linear state feedback design for surge control of axial flow compressor dynamics," *Journal of Marine Science and Technology*, vol. 22, 2014, pp. 352-361.

[11] D.-C. Liaw, and J.–T. Huang, "Fuzzy control for stall recovery of axial-flow compressor dynamics," *Journal of Control Systems and Technology*, vol. 6, 1998, pp. 231-241.

[12] D.-C. Liaw, and J.-T. Huang,, "Robust stabilization of axial flow compressor dynamics via sliding mode design," *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 123, 1998, pp. 488-495.

[13] D.-C. Liaw, C.-C. Song, and J.-T. Huang, "Robust stabilization of a centrifugal compressor with spool dynamics," *IEEE Trans. on Control Systems Technology*, vol. 12, 2004, pp. 966-972.

[14] J. S. Simon, and L. Valavani, "A Lyapunov based nonlinear control scheme for stabilizing a basic compression system using a close-coupled control valve," *Proc. 1991 IEEE American Control Conf.*, Boston, MA, June 26-28 1991, pp. 2398-2406.

[15] C. A. Mansoux, D. L. Gysling, J. D. Setiawan and J. D. Paduano, "Distributed nonlinear modeling and stability analysis of axial compressor stall and surge," *Proc. 1994 IEEE American Control Conf.*, Battimore, Maryland, June 1994, pp. 2305-2316.

[16] D.-C. Liaw, R. A. Adomatis, and E. H. Abed, "Two-parameter bifurcation analysis of axial flow compressor dynamics," *Proc. 1991 IEEE American Control Conf.*, Boston, MA, June 26-28 1991, pp. 2955-2960.

[17] D.-C. Liaw, W.-C. Lee, S.-M. Ren, and Y.-Y. Tsay, "A parametric study of axial flow compressor dynamics," *Proc. the 4th Pacific International Conference on Aerospace Science and Technology*, Kaohsiung Taiwan, 2001.

[18] E.J. Doedel, "AUTO: A program for the automatic bifurcation analysis of autonomous systems," *Congressus Numerantium*, vol. 30, 1981, pp. 265-284.

[19] H. K. Khalil, *Nonlinear Systems.* Pearson, 3rd ed., 2010.

[20] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations.* University of Louisville, Default Book Series, 1998, pp. 25-29.

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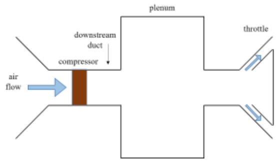


Fig. 1. Schematic diagram of compression system

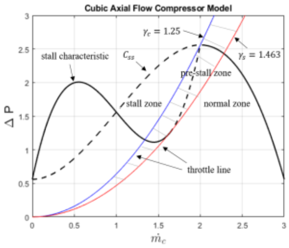


Fig. 2. A cubic axial flow compressor model defined in Eq. (22)

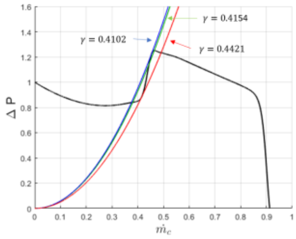
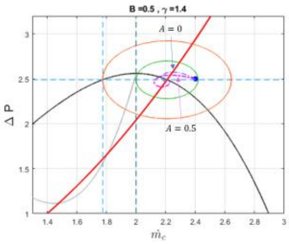


Fig. 3. The axial flow compressor model proposed in []



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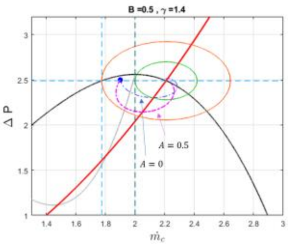

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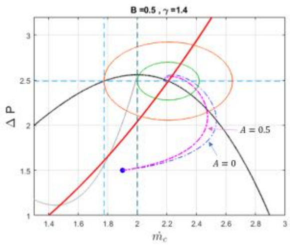
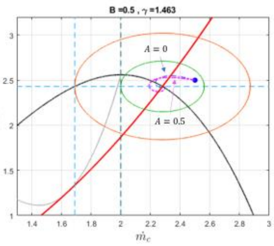


(c)

Fig. 4. Time response of ΔP vs. $\dot{m}_C$ in the pre-stall zone for $\gamma^0 = 1.4$: (a) $x(0) = (2.4, 2.5)$; (b) $x(0) = (1.9, 2.5)$; and (c) $x(0) = (1.9, 1.5)$.



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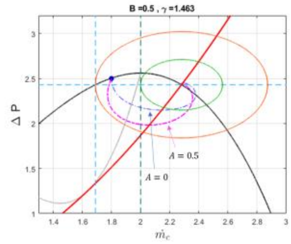

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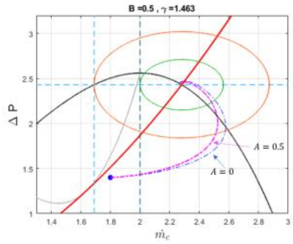
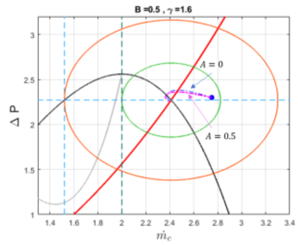


(c)

Fig. 5. Time response of ΔP vs. $\dot{m}_C$ in the pre-stall zone for $\gamma^0 = 1.463$: (a) $x(0) = (2.5, 2.5)$; (b) $x(0) = (1.8, 2.5)$; and (c) $x(0) = (1.8, 1.4)$.



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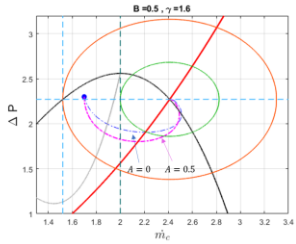

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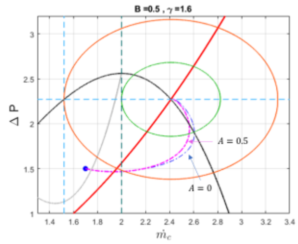


(c)

Fig. 6. Time response of ΔP vs. $\dot{m}_C$ in the normal un-stalled zone for $\gamma^0 = 1.6$: (a) $x(0) = (2.75, 2.3)$; (b) $x(0) = (1.7, 2.3)$; and (c) $x(0) = (1.7, 1.5)$.

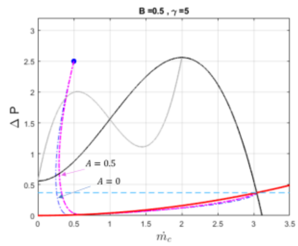
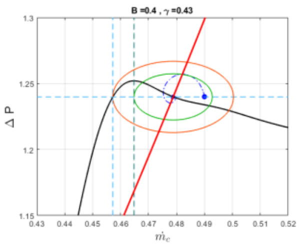


Fig. 7. Time response of ΔP vs. $\dot{m}_C$ in the normal un-stalled zone for $\gamma^0 = 5$: with initial $x(0) = (0.5, 2.5)$.



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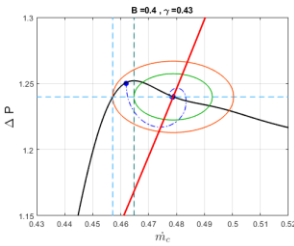

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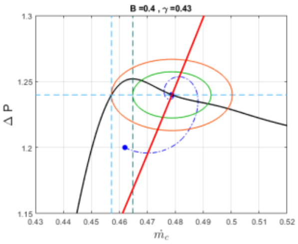

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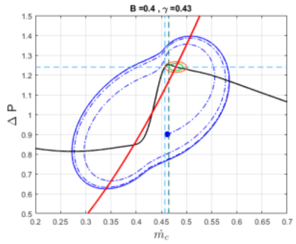

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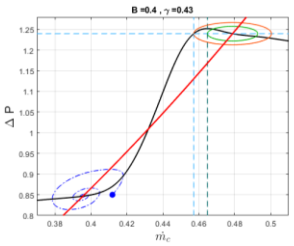
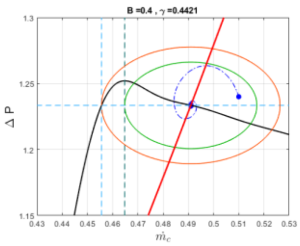


(e)

Fig. 8. Time response of ΔP vs. $\dot{m}_C$ in the pre-stall zone for $\gamma^0 = 0.43$: (a) $x(0) = (0.49, 1.24)$; (b) $x(0) = (0.462, 1.25)$; and (c) $x(0) = (0.462, 1.2)$; (d) $x(0) = (0.462, 0.9)$; and (e) $x(0) = (0.412, 0.85)$.



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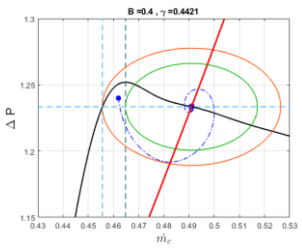

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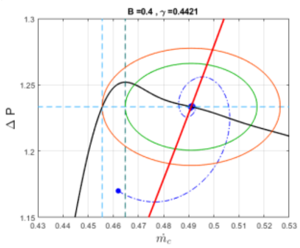
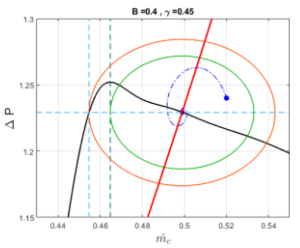


(c)

Fig. 9. Time response of ΔP vs. $\dot{m}_C$ in the pre-stall zone for $\gamma^0 = 0.4421$: (a) $x(0) = (0.51, 1.24)$; (b) $x(0) = (0.462, 1.24)$; and (c) $x(0) = (0.462, 1.17)$.



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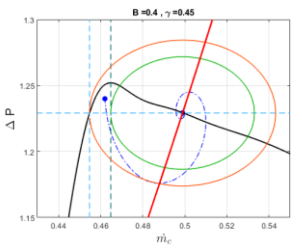

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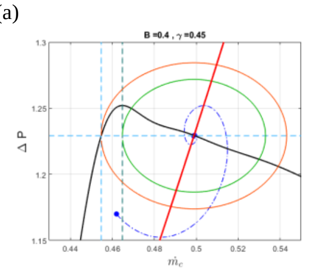

(c)

Fig. 10. Time response of ΔP vs. $\dot{m}_C$ in the normal un-stalled zone for $\gamma^0 = 0.45$: (a) $x(0) = (0.52, 1.24)$; (b) $x(0) = (0.462, 1.24)$; and (c) $x(0) = (0.462, 1.17)$.

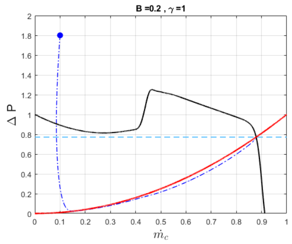

Fig. 11. Time response of ΔP vs. $\dot{m}_C$ in the normal un-stalled zone for $\gamma^0 = 1$: with initial $x(0) = (0.1, 1.8)$.