Relativistic Interpretation of Newton's Law of Universal Gravitation and Transition to Quantum Mechanics

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***Abstract*— In this paper it is shown a relativistic approach to describe body motion in a static gravitational field. Its description has been conducted for both inertial observer, and non-inertial observer. In this way, equations that determine apsidal precession in binary mass system and deflection of light in a gravitational field have been obtained. Furthermore, exponential metric has been derived from the use of Lagrange formalism to show its application in solving Dirac equation.**

***Index Terms*— Relativistic Interpretation, Universal Gravitation.**

I. Introduction

In classical mechanics, gravitational interaction is determined by Newton's law of universal gravitation that includes body mass as a fundamental property of matter; a body mass is a constant quantity and does not depend on its velocity. In relativistic mechanics, a body mass is not a constant quantity; it depends on its velocity in relation to an observer's system [1,2]. The relativistic description of gravitational interaction leads to the introduction of relativistic (effective) mass in Newton's law of universal gravitation [3]. By using this approach, possible implications of body motion in a static gravitational field will be considered.

Relativistic description of body movement in a static gravitational field

We will conduct relativistic description of body motion of mass m in a static gravitational field of mass source M, where $M >> m$, by using Einstein's relations from both general and special theory of relativity. For example, a body of mass m is in a static gravitational field of the source M whose strength is given by the following equation:

$$\vec{g} = -\frac{Rc^2}{|\vec{r}|^3}\vec{r} \quad (1)$$

where $R = \frac{GM}{c^2}$ is the gravitational radius. When a body of mass m_0 is moving at speed v_m it occurs a relativistic change of mass what is given by the following source:

$$m = \frac{m_0}{\sqrt{1-\left(\frac{v_m}{c}\right)^2}} \quad (2)$$

According to the relation we can consider the mass m as an effective body mass in a gravitational field that we will use in Newton's law of universal gravitation

$$\vec{F} = -\frac{Rc^2}{|\vec{r}|^3}m\vec{r} = -\frac{\partial U}{\partial \vec{r}} \quad (3)$$

where $\vec{r} = \vec{r}_0|\vec{r}|$, $U(r)$ - gravitional potential energy

Using Newton's law (3), body motion of mass m in a static gravitational field of the source mass M, will be considered. Description of the motion will be obtained from inertial system *S* and non-inertial system *S'* (Figure 1) [3].

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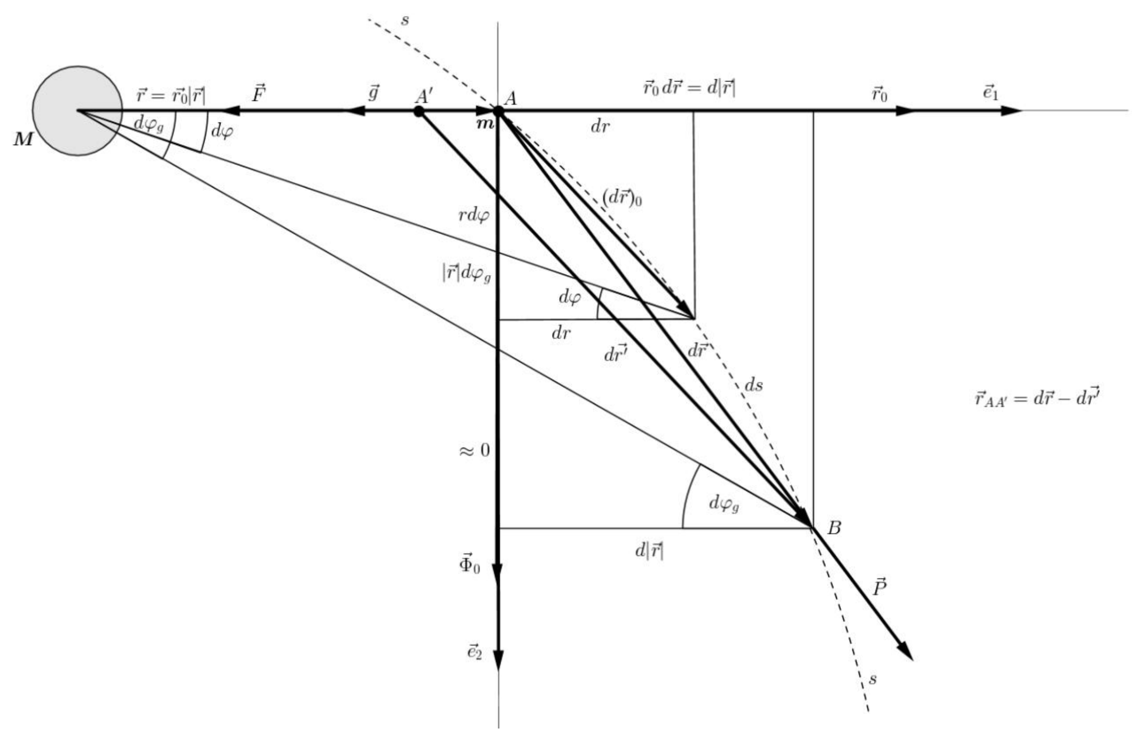

Figure 1

Since Reiman space is locally Euclidean, for an observer from the system S, the infinitesimal environment of the body mass m is Euclidean space.

The corresponding coordinate system is Minkowski's four-dimensional system with its components $(x^{\alpha=1,2,3}, x^0 = cT)$ and basis vector $\vec{\imath}_k$ for what the following applies:

$$\vec{\imath}_k \cdot \vec{\imath}_l = \mu_{kl} = \begin{Bmatrix} 1 & k = l = 1,2,3 \\ -1 & k = l = 0 \\ 0 & k \neq l \end{Bmatrix} \quad (4)$$

Four-vector $d\vec{s}$ in Minkowski's space is $d\vec{s} = \vec{\imath}_k dx^k,\ k = 0,1,2,3.$

The square of the arc element in this space will be:

$$-ds^2 = \mu_{kl} dx^k dx^l \quad (5)$$

Changing ili switching to polar coordinate system, the equation (5) becomes

$$-ds^2 = d|\vec{r}|^2 + |\vec{r}|^2 d\varphi^2 - c^2 dT^2 \quad (6)$$

where $\theta = \frac{\pi}{2} = const.$

The work of force $\vec{F}$ in a gravitational field is equal to the change of gravitational potential energy

$$dU = -\vec{F} d\vec{r} = \frac{Rc^2 m}{|\vec{r}|^2} d|\vec{r}| \quad (7)$$

It is known that

$$dE_k = -dU = c^2 dm \quad (8)$$

from the equations (7) and (8) differential equation that shows the change of body mass in a gravitational field follows, i.e.

,

$$\frac{dm}{m} = -\frac{R}{|\vec{r}|^2} d|\vec{r}|,\ R = const. \quad (9)$$

By integrating the equation we get the body mass in a gravitational field

$$m = m_\infty e^{\frac{R}{|\vec{r}|}} (10)$$

where $m = m_\infty$ when $|\vec{r}| \to \infty$.

If the mass from the equation (10) is introduced in the equation (7) a gravitational potential energy in the following form is obtained

$$U(\vec{r}) = -mc^2\left(1 - e^{-R/|\vec{r}|}\right) \quad (11)$$

while the total energy is

$$E_T = E_k + U = mc^2 e^{-R/|\vec{r}|} \quad (12)$$

Starting from equation (10) and using the expression for the equivalence of sluggish and heavy mass will be:

$$m = m_\infty e^{R/|\vec{r}|} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad (13)$$

Multiplying by c^2 and squaring equation (13) we get:

$$m_\infty^2 e^{2R/|\vec{r}|} c^4 = m_\infty^2 e^{2R/|\vec{r}|} v^2 c^2 + m_0^2 c^4$$

in fact, after a short edit, we get the expression for free energy in the form:

$$E = \sqrt{\vec{p}^2 c^2 + m_0^2 e^4} \quad (14)$$

$$E = m_\infty e^{R/|\vec{r}|} c^2 = mc^2 \quad (15)$$

$$\vec{p} = m_\infty e^{R/|\vec{r}|} v = mv$$

In inertial system S the Second Law of Motion

$$\vec{F} = \frac{d\vec{p}}{dT} \quad (16)$$

where

$$\vec{p} = m\frac{d\vec{r}}{dT} = m\vec{\imath}_\alpha \frac{dx^\alpha}{dT} = \vec{\imath}_\alpha p^\alpha; \alpha = 1,2,3 \quad (17)$$

From the equation we have

$$\frac{dm}{dT} = -\frac{R}{|\vec{r}|^2}\frac{d|\vec{r}|}{dT} m \quad (18)$$

Using expressions (17) and the equation becomes

$$\vec{F} = -m\frac{R}{|\vec{r}|^2}\frac{d|\vec{r}|^2}{dT^2} + m\frac{d^2|\vec{r}|}{dT^2} \quad (19)$$

By connecting the relations (19) and (3) with the replacement of variables $|\vec{r}| = \frac{1}{\rho}$ we get a relativistic equation of body motion in Minkowski's space

$$\ddot{\rho} + \rho = p^{-1} + R(\rho^2 + \dot{\rho}^2) \quad (20)$$

where

$$p^{-1} = \frac{Rc^2 m_0^2}{p_\varphi^2}, m = \frac{m_0}{\sqrt{1 - \frac{p_\varphi^2}{m^2 c^2}(\rho^2 + \dot{\rho}^2)}} \quad (21)$$

Non-inertial observer of system *S'* (Einstein's lift) assumes that every dot in a system *S'* has its corresponding dot in a non-inertial system *S*, i.e., bijection between coordinates of both systems

$$x^k = x^k(q^l) \quad (22)$$

where $(q^{\alpha=1,2,3}, q^0 = ct)$ are Gauss' coordinates.

Four-vector shift is

$$d\vec{s} = \vec{\imath}_k dx^k = \vec{\imath}_k \frac{\partial x^k}{\partial q^l} dq^l = \vec{e}_l dq^l \quad (23)$$

where

$$\vec{e}_l = \vec{\imath}_k \frac{\partial x^k}{\partial q^l} \quad (24)$$

is basis of Gauss coordinate system.

For basis vectors in Gauss' coordinate system relations these are valid:

$$\vec{e}_l \cdot \vec{e}_k = g_{lk};$$

$\vec{e}_l \cdot \overleftarrow{e}^k = \delta_{lk}$ (25)

$\vec{e}_l = g_{lk}\overleftarrow{e}^k$

The square of the element of shift will be:

$-ds^2 = \mu_{kl}dx^k dx^l = g_{lk}dq^l dq^k$ (26)

Using the similarity of triangles in system S $(d|\vec{r}|, d\varphi_g,\ cdT)$ and system S' $(dr,\ d\varphi,\ cdt)$, (Figure 1) by transformation into a polar coordinate system, we write the equation (26) in the form

$-ds^2 = g_{11}dr^2 + g_{22}d\varphi^2 + g_{00}c^2dt^2$ (27)

where

$$g_{00} = -\left(\frac{\partial T}{\partial t}\right)^2$$

$$g_{11} = \left(\frac{\partial|\vec{r}|}{\partial r}\right)^2 \quad (28)$$

$$g_{22} = |\vec{r}|^2\left(\frac{\partial\varphi_g}{\partial\varphi}\right)^2$$

Metric tensor g_{lk} is diagonal and its components are determined by using variational principle, i.e.

$dS = Ldt = \left(p_\alpha\frac{dq^\alpha}{dt} - H\right)dt = p_\alpha dq^\alpha + p_0 dq^0$ (29)

Four impulse $p = (p^{\alpha=1,2,3}, p^0)$ is

$$p^k = m_0 c\frac{dq^k}{ds} = m_0 c\frac{dq^k}{\sqrt{-g_{00}}dq^0\sqrt{1+g\alpha\beta\frac{dq^\alpha dq^\beta}{c^2dt^2}g^{00}}} = \left(m\frac{dq^\alpha}{dt}\sqrt{-g^{00}},\ mc\sqrt{-g^{00}}\right)$$

Since the Hamiltonian is equal to total energy according to the expression we get

$p_0 = -\frac{E}{c} = -mce^{-R/r} = g_{00}p^0 = g_{00}mc\sqrt{-g^{00}} = \sqrt{-g_{00}}mc$ (30)

From the equation is obtained

$g_{00} = -e^{-2R/r}$ (31)

Component of the metric tensor will be

$$From\ [6](8,2,6) \rightarrow\ \ g_{11} = -\frac{1}{g_{00}} = e^{2R/r}$$

$$From\ [3](16) \rightarrow g_{22} = r^2 g_{11} = r^2 e^{2R/r} \quad (32)$$

$$g_{33} = r^2 sin^2\vartheta e^{2R/r}$$

This way we have gotten a diagonal metric tensor of exponential metric [7]

$$g_{lk} = \begin{pmatrix} e^{2R/r} & 0 & 0 & 0 \\ 0 & r^2e^{2R/r} & 0 & 0 \\ 0 & 0 & r^2sin^2\vartheta e^{2R/r} & 0 \\ 0 & 0 & 0 & -e^{-2R/r} \end{pmatrix} \quad (33)$$

Component of force in basis $\vec{e}_\alpha$ is:

$F_\alpha = \vec{F}\ \cdot \vec{e}_\alpha = \vec{e}_\alpha\frac{d\vec{p}}{d\vec{t}} = (\frac{dp_\alpha}{dt} - \Gamma^\nu_{\alpha\mu}p_\nu\frac{dq^\mu}{dt})\frac{dt}{dT}$ (34)

We have $\Gamma^\nu_{\alpha\mu}$ Christoffel's symbols.

Covariant component of force [3] is:

$F_1 = -\frac{mRc^2}{r^2}$ (35)

$$\vec{F}d\vec{r} = inv = -\frac{Rc^2m}{|\vec{r}|^2}d|\vec{r}| = -\frac{Rc^2m}{r^2}dr \Rightarrow \frac{d|\vec{r}|}{dr} = \sqrt{g_{11}} = \frac{|\vec{r}|^2}{r^2} \quad (36)$$

If we equate the force from equation (34) with the force from Newton's law (3) we obtain the equation of motion of a body in a non-inertial system [4] with $m = m_\infty e^{\frac{R}{r}}$

Relativistic equation of body motion in Gauss space

$$\ddot{\rho} + \rho = p^{-1} + 4R\rho p^{-1} + R(\rho^2 + \dot{\rho}\quad^2) \quad (37)$$

where is $\dot{\rho} = \frac{d\rho}{d\varphi}$

The solution of equation (37) is for the Earth-bound observer from (32) [4] $\rho(\varphi) = p^{-}1(1 + ecos(1 - 3Rp^{-}1)\varphi) + \frac{3}{2}Rp^{-2}ecos\varphi$ which gives the perihelion displacement $\Delta = 6Rp^{-1}\pi = 42"$ at 100 years as opposed to equation (17) [4] for the Sun observer, which gives three times smaller the perihelion displacement. The deflection of light is the same for both observers $\Delta = 1.75"$.

II. Application of Exponential Metrics in Quantum Mechanics

The relativistic approach to describing the motion of a body in a gravitational field has led us to exponential metrics, as natural metrics, which are inherent in gravitational interactions. In the continuation of this paper we will show the application of this metric in the field of quantum mechanics in solving Dirac equation. Starting from the total energy according to equation (12) and using the relativistic relations between the energy and momentum of the particle of the Hamiltonian system, we write in the form:

$$H = mc^2 e^{\frac{-R}{r}} = e^{\frac{-R}{r}} c\sqrt{g^{lk}p_l p_k + m^2c^2} \qquad k,l = 1,2,3 \quad (38)$$

is g^{lk} exponential metric tensor per relation (33)

In Dirac approach, the Hamiltonian is linearly dependent on the momentum particle so equation (38) written in the form:

$$H = e^{\frac{-R}{r}}(c\vec{\alpha}\cdot\vec{p} + \beta mc^2) \quad (39)$$

where is $\vec{\alpha}$ and β matrices while $\vec{p}$ is the vector momentum of a particle of mass m.

Comparing the equations (38) and (39) we get the relation between the matrices $\vec{\alpha}$ and β and their connection with the impulse and the metric tensor, ie.

$$\{\alpha_l, \alpha_k\} = 2g^{lk} \cdot 1\delta_{lk}$$

$$\{\alpha_l, \beta\} = 0 \quad (40)$$

$$[p_l, \alpha^k] = 0$$

$$\alpha_l^2 = \beta^2 = 1$$

$$\alpha_l(r) = 2\sqrt{g^{lk}}\alpha_D^l$$

where α_D^l Dirac matrice, while δ_{lk} is Kronecker's symbol.

The effect of Hamiltonians (39) on the wave function psi will be shown by equation [5], respectively.

The action of the Hamiltonians (39) on real functiona Ψ we show by equation [5]

$$H\Psi = e^{\frac{-R}{r}}c(\vec{\alpha}\vec{p} + \beta mc)\Psi = E\Psi \quad (41)$$

where

$$\vec{\alpha}\cdot\vec{p}\Psi = (\frac{E}{c}e^{\frac{-R}{r}} - \beta mc)\Psi \quad (42)$$

In (39) an expression should be formulated $\vec{\alpha}\cdot\vec{p}$ Starting with $(\vec{\alpha}\cdot\vec{r})(\vec{\alpha}\cdot\vec{p}) = \vec{r}\cdot\vec{p} + i\vec{\sigma}\cdot\vec{L}$ where $\vec{\sigma}$ Pauli matrix

and $\vec{L}$ is angular moment.
Radius, shift and momentum is:

$$\vec{r} = \vec{e}_\alpha t^\alpha(q^\beta); \quad d\ r = \vec{e}_\alpha dq^\alpha \Rightarrow \frac{\partial t_\alpha}{\partial q^\beta} + \Gamma^\alpha_{\beta\gamma} t^\gamma = \delta_{\alpha\beta}; \quad p = \vec{e}_\alpha p^\alpha$$

For centrally symmetric potential is $t^3 = t^2 = 0$ we get

$$(\vec{\alpha}\cdot\vec{p}) = \frac{\alpha_r}{|\vec{r}|}\left(-ih\nabla_1 t^1 + i\left(h + \vec{\nabla}\cdot\vec{L}\right)\right) \quad (43)$$

$$(\vec{\alpha}_r) = \frac{\vec{\alpha}\cdot\vec{r}}{|\vec{r}|}$$

$\nabla_1 = \delta_1 + \Gamma$ - covariant derivation
Solutions of equation are sought in the form:

$$\Psi = \frac{1}{r}\begin{pmatrix} F & Y_+ \\ iG & Y_- \end{pmatrix} (44)$$

F and G – radial component of Ψ Y_+ and Y_- - angle component of Ψ

$$\alpha_r = \begin{pmatrix} 0 & \frac{\vec{\sigma}\cdot\vec{r}}{|\vec{r}|}\cdot \\ \frac{\vec{\sigma}\cdot\vec{r}}{|\vec{r}|} & 0 \end{pmatrix}; \vec{\alpha}\cdot\vec{r} = inv$$

$$\frac{|\vec{r}|^2}{r^2} = \sqrt{g_{11}} \quad (45)$$

$$\frac{\vec{\sigma}\cdot\vec{r}}{|\vec{r}|} Y_\pm = -Y_\mp; e^{\frac{-R}{r}} \frac{\frac{I+1}{2}}{r} \sim \frac{\frac{I+1}{2}}{r}$$

$$\left(h + \vec{\delta}\ \cdot\vec{L}\right)\Psi = -\left(I + \frac{1}{2}\right)\beta\Psi; \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

After correction (37) it becomes:

$$ih\nabla_1 \cdot \begin{pmatrix} iG \\ F \end{pmatrix} - ih\frac{I+\frac{1}{2}}{r}\begin{pmatrix} iG \\ -F \end{pmatrix} = \left(\frac{E}{c}e^{\frac{2R}{r}} - \beta e^{\frac{2R}{r}} m_0 c\right)\begin{pmatrix} F \\ iG \end{pmatrix} (46)$$

$\nabla_1 G$ and $\nabla_1 F$ are covariant derivations:

$$\nabla_1 G = \nabla_1 \Psi^2 = \frac{\partial G}{\partial r} + \Gamma^2_{12} G$$

$$\nabla_1 F = \nabla_1 \Psi^1 = \frac{\partial F}{\partial r} + \Gamma^1_{11} F$$

$$\Gamma^2_{11} F = \frac{1}{r}\left(\left(1 - \frac{R}{r}\right); \Gamma^1_{11} = -\frac{R}{r^2} \sim 0\right.$$

$$-h\left(\frac{\partial F}{\partial r} - \frac{\frac{I-1}{2}}{r} G\right) = \left(\frac{E}{c} r e^{2R/r} \frac{2R}{r} m_0 c\right) F \, (47)$$

$$h\left(\frac{\partial G}{\partial r} + \frac{\frac{I-1}{2}}{r} F\right) = \left(\frac{E}{c} r e^{2R/r} \frac{2R}{r} m_0 c\right) G$$

We develop the right side in a row, ignoring the members with a power of r greater than 1.

$$\alpha\left(\frac{\partial F}{\partial r}+\frac{\frac{I+1}{2}}{\rho}F\right)=\left(m_0+E+\frac{\alpha R}{\rho}(m_0+2E)\right)G \quad (48)$$

$$\alpha\left(\frac{\partial G}{\partial r}-\frac{\frac{I-1}{2}}{\rho}G\right)=\left(m_0-E+\frac{\alpha R}{\rho}(m_0-2E)\right)F$$

We are looking for a solution in the form:

$$F(\rho)=\rho^s e^{-\rho}\sum_{n<n'} a_n\rho^n \quad (49)$$

$$G(\rho)=\rho^s e^{-\rho}\sum_{n<n'} b_n\rho^n$$

regular in $\rho=0$ and $\rho=\infty$

Comparing by coefficients $n=n^{'}, n=n^{'}-1$ and with ρ^{-1} we get:

$$\alpha^2={m_0}^2-E^2$$

$$R^2({m_0}^2-4E^2)-\left(s+\frac{1}{2}\right)^2+I^2=0 \quad (50)$$

$$R^2({m_0}^2-2E^2)+\alpha\left(s+n^{'}+\frac{1}{2}\right)=0$$

With elimination s we get the bicubic equation with the variable

$$\alpha^2={m_0}^2-E^2, \beta=s+\frac{1}{2} \quad (51)$$

We get from [7]

$$4Rn'\alpha^3+({m_0}^2R^2+J^2-n')\alpha^2+2n'{m_0}^2R\alpha+{m_0}^4R^2=0 \quad (52)$$

$$2n'\beta^3+({m_0}^2R^2+J^2+n'^2)+2n'(3{m_0}^2R^2-J^2)\beta+n'^2(3{m_0}^2-J^2)\beta-({m_0}^2R^2-J^2)^2=0$$

The eigenvalue equation is bicubic which depends on the principal quantum number n. For the ground state, there is only one solution that corresponds to matter - antimatter. There are 3 realistic solutions for the offered conditions. In addition to matter - antimatter, there are two other dark substances - antimatter.

III. Conclusion

This paper shows a relativistic approach how to describe body motion in a static gravitational field. Gravitational interaction was realized by introducing a relativistic (effective) mass into Newton's law of universal gravitation. Furthermore, equations determining apsidal precession in binary systems and gravitational deflection of light have been obtained. With the use of Lagrange formalism, the exponential form of spacetime metric is derived. Exponential metric belongs to a group of alternative metrics and it confirms Einstein's theory of general relativity in the approximation of first member development in series. Application of this metric within Dirac quantum theory leads to the various solutions that describe gravitational quantum effects in the Planck wavelength domain.

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