International Journal of New Technology and Research (IJNTR)
ISSN: 2454-4116, Volume-5, Issue-3, March 2019 Pages 61-69

Note on the Null Controllability of Semi Linear Integro-Differential Systems in Banach Spaces with Distributed Delays in Control

Oraekie Paul Anaetodike

***Abstract*— Semi linear Integrodifferential Systems in Banach spaces with Distributed Delays in Control of the form**

$$\dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) + \int_{-\infty}^{t}\left(t, s, x_s(\phi)\right)$$

is presented for controllability analysis .Necessary and Sufficient Conditions for the systems to be null controllable are established. Uses were made of the Unsymmetric Fubinis' theorem and some Controllability Standards. The mild solution of the system was obtained using the variation of constant formula. From this mild solution, we extracted the set functions upon which our studies hinged.

***Index Terms*— Distributed Delays, Null-controllability, Semi linear, Integrodifferential Systems, Set functions.**

I. Introduction

Highlight According to**Oraekie (2018),** Neutral functional differential equations are characterized by a delay in the derivative of the form

$$\frac{d}{dt}[x(t) - Gx(t-h)] = Ax(t) + Bu(t)$$

Where $[-h, 0]$ is the delay interval , and
x an element of the Euclidean space E^n of
n − dimensions. A. G are nxnconstant matices ,
B is an nxm constant matrix and $h > 0$
real number. Equations of this form have applications in the study of electrical networks containing lossless transmission lines **(Bray ton (1976)** ,vibrational problems **(Ekgoltz(1964),**

Electrodynamics **(Driver (1963).**

One of the celebrated triumphs of La Salle was his solution of the null controllability problem of linear ordinary differential control system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1.1)$$

Where A is nxn constant matrix and B is an nxm constant matrix, when the controls are square integrable functions and lie in the unit cube:

$$C^m = \{u_j \in R^m : |u_j| \le 1 \; ; j = 1,2, \ldots, n\} \quad (1.2)$$

Where u_j denotes the jth component of U.

*He showed in his work(**La Salle** (**1959**)*

That if system (1.1) is proper (and this holds if and only if

$$rank[B, AB, A^2B, \ldots, A^{n-1}B] = n \quad (1.3)$$

And if the system

$$\dot{x}(t) = A(t)x(t) \quad (1.4)$$

is stable(i.e. all the eigenvalues of A have no positive real part) ,then system(1.1) is null controllable with constraints.

The rank condition in system (1.3) is equivalent to the controllability of system (1.1) when the controls are ''big'' in the sense that they are only assumed to be square integrable. This is equivalent to null controllability with square integrable controls. We call such controls unrestrained in contrast to the restrained controls which lie in a closed and bounded set**(Oraekie (2018)).** But for delay systems, null controllability is not equivalent to controllability. For instant, all ntth order scalar differential difference equations of retarded type are null controllable **(H.T.Banks, M.Q.Jacobs and C.E.Langenhop (1975)),** where as they are never controllable. For the delay system of the form

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) \;, t \ge \partial \quad (1.5)$$

$$x_\partial = \phi \in W_2^{(1)}([-h, 0]) \equiv W_2^{(1)}$$

Chukwu (1984), proved that if the system (1.5) is controllable with unrestrained controls, and if

$$\dot{x}(t) = L(t, x_t) \;, t \ge \partial \quad (1.6)$$

is uniformly asymptotically stable, then system(1.5) is null controllable with constrained controls. The problem was posed on whether the weaker condition of null controllability with unrestrained controls and the uniformly asymptotic stability assumption was sufficient for restrained null controllability. The issue was settled in **Chukwu (19 97)** .That is, it was shown that if the system (1.5) $(\boldsymbol{i.e.}, \dot{\boldsymbol{x}}(\boldsymbol{t}) =$ ***Lt ,xt+Btut,*** is null controllable with square integrable controls and if the system(1.6)uniformly asymptotically stable, then system(1.5 is null controllable with square integrable controls which lie in a closed unit ball with zero in its interior.

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional spaces has been extensively studied. Several authors have extended the concept to infinite dimensional systems represented by the evolution equations with bounded operators in Banach spaces**(Oraekie(2016) ,Naito(1992)).**Recently, **Oraekie(2017)** established

Oraekie, Paul Anaetodike, Department Of Mathematics, Faculty Of Physical Sciences Chukwuemeka Odumegwu Ojukwu University, Uli-Campus Anambra State, Nigeria, West Africa.

necessary and sufficient conditions for null controllability of nonlinear infinite dimensional space of neutral differential systems with distributed delays in the control.

The purpose of thiswork,therefore, is to extend the concept of null controllability with constrained controls to the systems of Semi linear Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control.

II. PRELIMINARIES / NOTATIONS

2.1. DESCRIPTION OF SYSTEM

Our specific objective is to study the controllability the Semi linear Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control of the form.

$$\dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) + \int_{-\infty}^{t} f\big(t,s,x_s(\phi)\big)\ ; t \in J = [t_0,t_1] \quad (2.1)$$

$x_t(\phi) = \phi(t); t \in (-\infty, 0]$

Through its semi linear base control system

$$\dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) \quad (2.2)$$

and its free system

$$\dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-\infty}^{t} f\big(t,s,x_s(\phi)\big) \quad (2.3)$$

Here, $\{A(t): t \geq 0\}$ *is a family of bounded linear operators mapping a Banach space* X *to* X. *The state* $x(t)$ *takes values in the Banach space* X *and the control function* u *is given* $L_2(J, U)$, *a Banach space of admissible control functions with* U *as a Banach space.* $H(t,\theta)$ *is an nxm matrix continuous at and of bounded variation in* θ *on* $[-h,0]; h > 0$ *for each* $t \in J = [t_0,t_1], t_1 > 0 = 0.$

Let X_α *denote the interpolation space defined in the* α *power of* $A(o)$ *that is,*

$$X_\alpha = \big[x : x \in D\big(A^\alpha(0)\big)\big] \text{ with } \|x\|_\alpha = \|A^\alpha(0)\|$$

The space C_α *is the space of bounded, uniformly continuous function* ϕ *from* $(-\infty,0]$ *to* X_α *Endowed with the supremum norm:* $\|\phi\|_{C_\alpha} = sup\{\|\phi(\theta)\|_\alpha : \theta \in (-\infty,0]\}$.

Furthermore, let $\phi \in C_\alpha$ *for some* $\alpha \in (0,1)$, *and* f *is a continuous nonlinear operator of* $JxJxX_\alpha$ *into* X. *For the existence of a solution of system* (2.1), *we use the following assumptions as contained in* ***Dauer and Balasubramaniam*** **(1997)**:

(1). *The domain* $D(A)$ *of* $A(t), t \in J = [t_0,t_1], t_1 > 0$, *is dense in the Banach space* X *and independent of* t.

(ii). For each $t \in [0,\infty)$, *the resolvent* $R\big(\lambda, A(t)\big)$ *exists for all* λ *such that* $Re\lambda \geq 0$, *and there exists* $C > 0$ *such that* $\big\| R\big(\lambda, A(t)\big)\big\| \leq \dfrac{C}{|\lambda|+1}$.

(iii). For any $t,s,\tau \in J = [t_0,t_1]$, *there exists a* $\delta > 0$ *such that* $\delta \in (0,1)$ *and* $K > 0 \ni$

$$\big\|\big(A(t) - A(\tau)\big)A^{-1}(s)\big\| \leq K|t-\tau|^\delta$$

And for each $t \in [t_0,t_1]$ *and some* $\lambda \in \rho\big(A(t)\big)$, *the resolvent* $R\big(\lambda, A(t)\big)$ *set of* $A(t)$ *is compact operator. The fact that* $0 \in \rho\big(A(t)\big)$ *and* $-A(r)$ *generates an analytic semigroup implies that fractional powers of* $A(r)$ can be defined for $0 < \alpha < 1$. *We put*

$$A^{-\alpha}(r) = \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}e^{-sA(r)}ds,$$

where $\Gamma(.)$ *denotes the Eulerian gamma function. The operator* $A^{-\alpha}(r)$ can be shown to be a bounded linear operator with well defined inverse(***Rankin*** **(1982)**). **Friedman (1969)** established that if conditions $(i) - (iii)$ above are satisfied, then there exists an operator valued function $\mathrm{X}(t,\tau)$ which is defined on the triagle $0 \leq \tau \leq t < \infty$. $X(t,\tau)$ is strongly continuous jointly in t and τ, maps X into $D(A)$ if $t > \tau$. *The family* $\{\mathrm{X}(t,\tau) : 0 \leq \tau \leq t < \infty\}$ satisfies the identity

$$\mathrm{X}(t,\tau) = \mathrm{X}(t,s)\mathrm{X}(s,\tau): 0 \leq \tau \leq t < \infty.$$

The derivative $\dfrac{\delta}{\delta t}\mathrm{X}(t,\tau)$ existe in the strong operator topology and belongs to X whenever $0 \leq \tau < t$ (***Pazy*** **(1982)**).

Finally, $\mathrm{X}(t,\tau)$ satisfies the following initial value problem

$$\frac{\delta}{\delta t}\mathrm{X}(t,\tau)=\mathrm{A}(t)\mathrm{X}(t,\tau),\text{for } t>\tau$$
$$\mathrm{X}(\tau,\tau)=\mathrm{I}\text{ , whenI is the identity operator.}$$

Furthermore $\left(\textbf{\textit{see Fitzgibbon}}\ (\mathbf{1990})\right)$, if $0\leq \mathrm{v}\leq 1\,;\ 0\leq\beta\leq\delta<1+\mu$, *then for any* $0\leq\tau\leq \mathrm{t}+\Delta \mathrm{t}<t_1$ *and* $s\in J$, *there exists a* $K(\beta,v,\delta)$ such that

$$\left\|A^{v}(s)\left[X(t+\Delta t,\tau)-\mathrm{X}(t,\tau)A^{-\beta}(r)\right]\right\|\leq K(\beta,v,\delta)(\Delta t)^{\delta-V}|t-\tau|^{\beta-\delta}\,(2.4)$$

(iv) . *There exists a* $\beta_0>\alpha$ *and* $\omega>0$ *such that for* $0\leq\beta<\beta_0$, *there exists a* $K_\beta>0$ *satisfying*

$$\left\|A^{\beta}(0)X(t,s)\right\|\leq K_\beta(t-s)^{-\beta}e^{-\omega(t-s)}\,(2.5)$$

(v). *The function* $f\colon JxJxX_\alpha\to X$ *is continuous* , $f(t,s,0)=0$ *for* $s<t$, *and there exists an* $L>0$ *and* $v>0$ *such that*

$$\|f(t,s,x)-f(t,s,y)\|\leq e^{-v(t-s)}L\|x-y\|_\alpha\,(2.6)$$

(vi). *The bounded linear operator* $A^{-\alpha}(t)$ *is compact for all* $\alpha\in(0\,,1]$.
Suppose that $(i)-(iii)$ *and* (v) *are satisfied* .
If $\phi\in C_\alpha$ *and* $\phi(0)\in D\left(A^{\beta}(0)\right)$ *for some* $\beta>\alpha$ *and* (iv) *is satisfied for some* $\beta_0>\beta$, *then there exists a unique function* $x_t(\phi)\,:\,J\to X$ *such that*

$$x_t(\phi)=X(t,0)\phi(0)+\int_0^t X(t,s\,)\int_{-\infty}^{s} f\big(s,\tau,x_\tau(\phi)\big)\,d\tau ds$$
$$+\int_0^t X(t,s\,)\left[\int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta)\right]ds \qquad (2.7)$$
$$x_t(\phi)=\phi(t);t\in(-\infty,0].$$

Moreover, $x_t(\phi)$ *is continuously differentiabe for* $t>0$ *and satisfies system* (2.1)

2.3. *EXPLICT VARIATION OF CONSTANT FORMULA*

A careful observation of the system (2.7)shows that the values of the control u(t) for $t\in[-h,t]$enter the definition of the complete state $(\mathbf{z}(\mathbf{t_0})=\{\mathbf{x},\mathbf{u_t}\})$ thereby creating the need for an explicit variation of constant formula. The control in the last term of the right hand side of the formula (2.7), therefore, has to be transformed by applying the method of Klamka as contained in **Klamka (1978)**. Firstly, we interchange the order of integration using unsymmetric Fubinis' theorem to have

$$x_t(\phi)=X(t,0)\phi(0)+\int_0^t X(t,s\,)\int_{-\infty}^{s} f\big(s,\tau,x_\tau(\phi)\big)\,d\tau ds$$
$$+\int_{-h}^{0} dH_\theta\int_{0+\theta}^{t+\theta} X(t,s-\theta\,)\,H(s-\theta,\theta)u(s+\theta-\theta)\,ds \qquad (2.8)$$

Simplifying system(2.8), *we have*

$$x_t(\phi)=X(t,0)\phi(0)+\int_0^t X(t,s\,)\int_{-\infty}^{s} f\big(s,\tau,x_\tau(\phi)\big)\,d\tau ds$$
$$+\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta\,)\,H(s-\theta,\theta)u_0(s)\,ds$$
$$+\int_{-h}^{0} dH_\theta\int_{\theta}^{t+\theta} X(t,s-\theta\,)\,H(s-\theta,\theta)u(s)\,ds \qquad (2.9)$$

Using again the Unsymmetric Fubinis' theorem on the change order of integration and incorporating $\boldsymbol{H}^*$ *as defined below,* $\boldsymbol{H}^*(\boldsymbol{s},\boldsymbol{\theta})=\begin{cases}\boldsymbol{H}(\boldsymbol{S},\boldsymbol{\theta}),\ \textbf{\textit{for } }\boldsymbol{s}\leq\boldsymbol{t}\\ \quad\mathbf{0}\ ,\textbf{\textit{for } }\boldsymbol{s}\geq\boldsymbol{t}\end{cases}$ (2.10)

System (2.9) *be comes*

$$x_t(\phi)=X(t,0)\phi(0)+\int_0^t X(t,s\,)\int_{-\infty}^{s} f\big(s,\tau,x_\tau(\phi)\big)\,d\tau ds$$
$$+\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta\,)\,H(s-\theta,\theta)u_0(s)\,ds+\int_0^t\int_{-h}^{0} X\,(t,s-\theta)d\theta\,H^*(s-\theta,\theta)u(s)\,ds \qquad (2.11)$$

Integration is still in the Lebesgue Stieltjes sense in the variable θ *in* H.

For brevity, let

$$\eta(t) = X(t,0)\phi(0) + \int_0^t X(t,s) \int_{-\infty}^{s} f\left(s,\tau,x_\tau(\phi)\right) d\tau ds \qquad (2.12)$$

$$\mu(t) = \int_{-h}^{0} dH_\theta \int_{\theta}^{0} X(t,s-\theta)\, H(s-\theta,\theta) u_0(s)\, ds \qquad (2.13)$$

$$Z(t,s) = \int_0^t \int_{-h}^{0} X(t,s-\theta) d\theta\, H^*(s-\theta,\theta) (2.14)$$

$$Z^T(t,s) = \left[\int_{-h}^{0} X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]^T (2.15)$$

Substituting systems (2.12), (2.13) *and* (2.14) *in system*(2.11), *we have explicit variation of constant formula for the system*(2.1) *as*

$$\boldsymbol{x_t(\phi) = \eta(t) + \mu(t) + z(t,s)u(s)\, ds} \qquad \mathbf{(2.16)}$$

2.4 . BASIC SET FUNCTIONS AND PROPERTIES

***Definition* 2.4.1(*Reachable set*)**

The reachable set of system(2.1) *is given as*

$$R(t,t_0) = \left\{\int_0^t \int_{-h}^{0} X(t,s-\theta) d\theta\, H^*(s-\theta,\theta) u(s)\, ds : u \in C^m \subset L_2(J,U) and\ |u_j| \le 1, \forall\, j. \right\}$$

***Definition* 2.4.2. (*Attainable set*)**

The Attainable setof system(2.1) *is given as*

$$A(t,t_0) = \{x_t(\phi) : u \in C^m \subset L_2(J,U) and\ |u_j| \le 1, \forall\, j\}$$

***Definition* 2.4.3. (*Target set*)**

The Target setof system(2.1) *is given as*

$$G(t,t_0) = \left\{\begin{matrix} x_t(\phi) : t \in J\, ; t \ge \mathrm{T} > t_0 = 0\ for\ some\ fixed\ \mathrm{T}\ and\ u \in C^m \subset L_2(J,U) \ni \\ |u_j| \le 1, \forall\, j \end{matrix}\right\}$$

***Definition* 2.4.4. (*Controllability Grammian*).**

TheControllability Grammianof system(2.1) *is given as*

$$W(t,t_0) = \int_0^t \left[\int_{=h}^{0} X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]\left[\int_{=h}^{0} X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]^T$$

where T denotes matrix transpose. Put

$$\boldsymbol{u(t) = -Z(t,s)W^{-1}(t,t_0)[x_t(\phi)]} \qquad \mathbf{(2.17)}$$

***Definition* 2.4.5. (*Properness*).**

The system(2.1) *is said to be proper on an interval* $[t_0,t_1]$ *if*

$$C^T\left[\int_{=h}^{0} X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right] = 0\ \ a.e\ \Rightarrow C = 0$$

That is, system(2.1)*is proper in* R^n *on*$[t_0,t_1]$ *if span* $R(t,t_0) = R^n$.

***Definition* 2.4.6. (*Complete Controllability*)**

The system(2.1) *is said to be Completely Controllableon an interval* $J - [t_0,t_1]$ *if for every continuous initial function* ϕ *and every state* $x_1 \in R^n$, *there exists an admissible control energy function* $u \in U$ *such that a solution of the system*(2.1) *satisfies*

$$\boldsymbol{x_t(\phi) = x_{t_1}(\phi).}$$

$\boldsymbol{Definition\ 2.4.6.}$ ($\boldsymbol{Null\ Controllability}$)
The system(2.1) is said to be NullControllableon an interval $J = [t_0, t_1]$ *if for every continuous initial function* $\phi \in C_\alpha$ *, there exists an admissible control energy function* $u \in U$ *such that a solution* $x_t(\phi)$ *of the system(2.1) satisfies*

$$\boldsymbol{x_{t_1}(\phi) = 0.}$$

III. MAIN RESULT

Consider the system(2.1)given below

$$\dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) + \int_{-\infty}^{t} f\big(t, s, x_s(\phi)\big)\ ; t \in J \qquad \big(system(2.1)\big)$$
$$x_t(\phi) = \phi(t); t \in (-\infty, 0]$$

Assume for system(2.1) that :

(i) *. that the constraint set* $U \subset L_2$ *is an arbitrary compact subset of* R^n.

(ii) *. The free system(2.3) given as*

$$\dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-\infty}^{t} f\big(t, s, x_s(\phi)\big)\ \big(system(2.3)\big)$$

is uniformly asymptotically stable sothat the solution of the system(2.3) satisfies exponential estimate. i.e.

$$\|x_t(\phi, t_0)\| \le Me^{q(t-t_0)}\|\phi\|\ , for\ a, M > 0\ ;\ t_0 = 0$$

(iii) *. The Semilinear controlsystem (2.2) is proper in* R^n.

$$i.e.\ \ \dot{x}_t(\phi) = -A(t)x_t(\phi) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta)\ \big(system(2.2)\big)$$

Then system(2.1) is null controllable.

Proof.

Recall that the controllability grammian denoted by $W(t, t_0)$*has an inverse and the invertibility of the grammian of any dynamical control system garantees the controllability of the system(* ***Oraekie*** **(2013)**). *Thus, by* (iii) $W^{-1}(t, t_0)$ *exists for each* $t > 0$.

Suppose that the pair of functions x, u *form a solution pair to the set of integral equations:*

$$\boldsymbol{u(t) = -Z^T(t,s)W^{-1}(t,t_0)}\left[\boldsymbol{x_t(\phi)} - \int_0^t \int_{-h}^{0} X\,(t, s-\theta)d\theta\, H^*(s-\theta,\theta)u(s)\,ds\right]$$

$$\Rightarrow \boldsymbol{u(t)} \ = \left(-\left[\int_0^t \int_{-h}^{0} X\,(t, s-\theta)d\theta\, H^*(s-\theta,\theta)\right]^T\right)$$
$$x\left(\int_0^t \left[\int_{=h}^{0} X(t, s-\theta)\, d\theta H^*(s-\theta,\theta)\right]\left[\int_{=h}^{0} X(t, s-\theta)\, d\theta H^*(s-\theta,\theta)\right]^T\right)^{-1}$$
$$\boldsymbol{x}\left[\boldsymbol{x_t(\phi)} - \int_0^t \int_{-h}^{0} X\,(t, s-\theta)d\theta\, H^*(s-\theta,\theta)u(s)\,ds\right]$$

$$\Rightarrow \boldsymbol{u(t)} = \left(-\left[\int_0^t \int_{-h}^0 X\,(t, s-\theta) d\theta\, H^*(s-\theta,\theta)\right]^T\right)\boldsymbol{x}$$

$$\left(\int_0^t \left[\int_{=h}^0 X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]\left[\int_{=h}^0 X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]^T\right)^{-1} x$$

$$[X(t,0)\phi(0) + \int_0^t X(t,s\,) \int_{-\infty}^s f\big(s,\tau,x_\tau(\phi)\big)\, d\tau ds$$

$$+ \int_{-h}^0 dH_\theta \int_\theta^0 X(t,s-\theta\,)\, H(s-\theta,\theta) u_0(s)\, ds + \int_0^t \int_{-h}^0 X\,(t,s-\theta) d\theta\, H^*(s-\theta,\theta) u(s)\, ds$$

$$-.\int_0^t \int_{-h}^0 X\,(t,s-\theta) d\theta\, H^*(s-\theta,\theta) u(s)\, ds\ ]$$

$$\Rightarrow \boldsymbol{u(t)} = \left(-\left[\int_0^t \int_{-h}^0 X\,(t, s-\theta) d\theta\, H^*(s-\theta,\theta)\right]^T\right)$$

$$x\left(\int_0^t \left[\int_{=h}^0 X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]\left[\int_{=h}^0 X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]^T\right)^{-1} x$$

$$\left[X(t,0)\phi(0) + \int_0^t X(t,s\,) \int_{-\infty}^s f\big(s,\tau,x_\tau(\phi)\big)\, d\tau ds + \int_{-h}^0 dH_\theta \int_\theta^0 X(t,s-\theta\,)\, H(s-\theta,\theta) u_0(s)\, ds\right] \quad (2.18)$$

For some suitable chosen $t_1 \geq \boldsymbol{t} \geq \boldsymbol{t_0} = \boldsymbol{0}$ *,* ***we have***

$$x_t(\phi) = X(t,0)\phi(0) + \int_0^t X(t,s\,) \int_{-\infty}^s f\big(s,\tau,x_\tau(\phi)\big)\, d\tau ds$$

$$+ \int_{-h}^0 dH_\theta \int_\theta^0 X(t,s-\theta\,)\, H(s-\theta,\theta) u_0(s)\, ds + \int_0^t \int_{-h}^0 X\,(t,s-\theta) d\theta\, H^*(s-\theta,\theta) u(s)\, ds \qquad (2.19)$$

$$x_t(\phi) = \phi(t)\,;\, t \in [t_0 - \lambda, t_0]$$

$$\Rightarrow x_{t_1}(\phi) = X(t,0)\phi(0) + \int_0^{t_1} X(t,s\,) \int_{-\infty}^{t_1} f\big(s,\tau,x_\tau(\phi)\big)\, d\tau ds$$

$$+ \int_{-h}^0 dH_\theta \int_\theta^0 X(t,s-\theta\,)\, H(s-\theta,\theta) u_0(s)\, ds$$

$$+ \int_0^{t_1} \int_{-h}^0 X\,(t,s-\theta) d\theta\, H^*(s-\theta,\theta) \left(-\left[\int_0^{t_1} \int_{-h}^0 X\,(t,s-\theta) d\theta\, H^*(s-\theta,\theta)\right]^T\right)$$

$$x\left(\int_0^{t_1} \left[\int_{=h}^0 X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]\left[\int_{=h}^0 X(t,s-\theta)\, d\theta H^*(s-\theta,\theta)\right]^T\right)^{-1} x$$

$$\left[X(t,0)\phi(0) + \int_0^{t_1} X(t,s\,) \int_{-\infty}^{t_1} f\big(s,\tau,x_\tau(\phi)\big)\, d\tau ds + \int_{-h}^0 dH_\theta \int_\theta^0 X(t,s-\theta\,)\, H(s-\theta,\theta) u_0(s)\, ds\right]\ ds$$

$$\Rightarrow\ x_{t_1}(\phi) = X(t,0)\phi(0) + \int_0^{t_1} X(t,s\,) \int_{-\infty}^{t_1} f\big(s,\tau,x_\tau(\phi)\big)\, d\tau ds$$

$$+\int_{-h}^{0} dH_\theta \int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds$$

$$-\frac{\left[\int_0^{t_1}\int_{-h}^{0} X(t,s-\theta)d\theta\, H^*(s-\theta,\theta)\right]^T\left(\int_0^{t_1}\int_{-h}^{0} X(t,s-\theta)d\theta\, H^*(s-\theta,\theta)\right)}{\int_0^{t_1}\left[\int_{=h}^{0} X(t,s-\theta)\,d\theta H^*(s-\theta,\theta)\right]\left[\int_{=h}^{0} X(t,s-\theta)\,d\theta H^*(s-\theta,\theta)\right]^T}\Bigg[X(t,0)\phi(0)$$

$$\left.+\int_0^{t_1} X(t,s)\int_{-\infty}^{t_1} f(s,\tau,x_\tau(\phi))\,d\tau ds+\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds\right]$$

$$\Rightarrow x_{t_1}(\phi)=X(t,0)\phi(0)+\int_0^{t_1} X(t,s)\int_{-\infty}^{t_1} f(s,\tau,x_\tau(\phi))\,d\tau ds$$

$$+\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds$$

$$-1\left[X(t,0)\phi(0)+\int_0^{t_1} X(t,s)\int_{-\infty}^{t_1} f(s,\tau,x_\tau(\phi))\,d\tau ds+\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds\right]$$

$$\Rightarrow x_{t_1}(\phi)=X(t,0)\phi(0)+\int_0^{t_1} X(t,s)\int_{-\infty}^{t_1} f(s,\tau,x_\tau(\phi))\,d\tau ds$$

$$+\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds-X(t,0)\phi(0)$$

$$-\int_0^{t_1} X(t,s)\int_{-\infty}^{t_1} f(s,\tau,x_\tau(\phi))\,d\tau ds-\int_{-h}^{0} dH_\theta\int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds = 0.$$

It remains to show that u is an admissible control energy function. That is we need to show that u is a function from the interval $J=[t_0,t_1]$ *to* U ($\boldsymbol{u}:[\boldsymbol{t_0},\boldsymbol{t_1}]\to\boldsymbol{U}$) *is in the arbitrary compact constraint subset of* R^m. *i.e.* $|u|\le r$ *for some* $r>0$ *and* $r\in(0,1)$.

By (ii)

$$|Z^T(t_1,s)W^{-1}(t_1,t_0)|\le b_0,\ for\ some\ b_0>0.$$

and

$$\left|X(t_1,t_0)x_{t_1}(0)\right|\le e^{-a(t_1,t_0)},\qquad for\ some\ b_1>0$$

Hence,

$$|u(t)|\le b_0\left[b_1 exp\left(-a(t_1,t_0)\right)\right]\int_{t_0}^{t_1} b_2\, exp\left(-a(t_1,s)\right)exp(-\rho s)$$

. Thus ,

$$|u(t)|\le b_0\left[b_1 exp\left(-a(t_1,t_0)\right)\right]\rho b_2 exp(-at_1)(2.19)$$

Since $\rho-a\ge 0$ *and* $s\ge t_0\ge 0$

Taking t_1 *sufficiently large, we have* $|u(t)|\le r$, $t\in[t_0,t_1]$, *showing that u is an admissible control.*

Next, we prove the existence of a solution pair of integral equations (2.18) and(2.19).

Let E be the Banach space of of all functions (x,v): $[t_0-h,t_1]\mathrm{x}[t_0-h,t_1]\to R^n\mathrm{x}R^m$, where x ∈ E$([t_0-h,t_1],R^n)$, u ∈ $L_2([t_0-h,t_1],R^m)$ with the norm defined by

$$\|(x,v)\|=\|x\|_2+\|x\|_2,$$

$$\text{where},\quad \|x\|_2=\left[\int_{t_0-h}^{t_1}|x(s)|^2\,ds\right]^{\frac{1}{2}};\ \|u\|_2=\left[\int_{t_0-h}^{t_1}|u(s)|^2\,ds\right]^{\frac{1}{2}}.$$

Define the operator $T\colon E\to E$ *by*

$$T(x,u)=(y,v), where$$

$$\boldsymbol{v(t)} = -\boldsymbol{Z^T(t,s)W^{-1}(t,t_0)}\left[X(t,0)\phi(0) + \int_0^{t_1} X(t,s)\int_{-\infty}^{s} f\big(s,\tau,x_\tau(\phi)\big)\,d\tau ds + \int_{-h}^{0} dH_\theta \int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds\right] (2.20)$$

$for\ t \in [t_0, t_1]\ , and\ \ v(t) = \omega(t)\,, t \in [t_0 - \lambda, t_0].$

$$y_t(\phi) = X(t,0)\phi(0) + \int_0^{t} X(t,s)\int_{-\infty}^{s} f\big(s,\tau,x_\tau(\phi)\big)\,d\tau ds + \int_{-h}^{0} dH_\theta \int_{\theta}^{0} X(t,s-\theta)\,H(s-\theta,\theta)u_0(s)\,ds + \int_0^{t}\int_{-h}^{0} X(t,s-\theta)d\theta\, H^*(s-\theta,\theta)u(s)\,ds \qquad (2.21)$$

$for\ t \in [t_0, t_1]\ , and\ \ y_t(\phi) = \phi(t)\,; t \in [t_0 - \lambda, t_0].$

$We\ have\ already\ shown\ above\ \ that\ |u(t)| \le r\,, t \in J = [t_0, t_1]\ and\ v:\ [t_0 - h, t_1] \to U,$ $\boldsymbol{we}\ |v(t)| \le r\,.$

$$Hence, \|\boldsymbol{v(t)}\|_2 \le \boldsymbol{r(t_1 + h - t_0)^{\frac{1}{2}} = c_0}.$$

$Again\,, |y_t(\phi)| < b_1 exp[-a(t-t_0)] + b_3 \int_{t_0}^{t} |v(s)|\,ds + \rho b_2 exp(-at_1)$

$b_3 = sup|Z(t,s)|. Since\ a > 0\,, t \ge t_0 \ge 0\,, we\ conclude\ that$

$$|\boldsymbol{y_t(\phi)}| \le \boldsymbol{b_1 + b_3 a(t-t_0) + \rho b_2 = c_1}\,, t \in J, and$$

$|\boldsymbol{y_t(\phi)}| \le \boldsymbol{sup|\phi(t)| = d\,, t \in [t_0 - \lambda, t_0]}$

$\boldsymbol{Hence, if\ n = max(c_1, d)\,, then}$

$$\|\boldsymbol{y}\|_2 \le \boldsymbol{n(t_1 + h - t_0)^{\frac{1}{2}} = c_2} < \infty, t \in J$$

$Let\ k = max(c_0, c_2)\,. Then\,, if\ we\ let\ B(l) = \{(x,u) \in E:\ \|x\|_2 \le l\,; \|y\|_2 \le l\}$

$We\ have\ shown\ thus\ that\ T: B(l) \to B(l).$

$Since\ B(l)\ is\ closed, bounded\ and\ convex, by\ Riesz\ theorem($ $\boldsymbol{See\ L.V.Kantorovich\ and\ G.P.\ Akilov(1982)}$, $Functional\ Analysis, Pergamon\ Press, Oxford),$ $\boldsymbol{Onwuatu(1993)}$, $KYBERNRTIKA, VOL29, N04, PP325-336,$ $\boldsymbol{Oraekie(2018)}$, $)\ it\ is\ relatively\ compact\ under\ the\ transformation. Hence, system(2.1)\ is\ null\ controllable.$

IV. CONCLUTION

In this work, necessary and sufficient conditions for the Semi linear IntegrodifferentialSystems in Banach spaces with Distributed Delays in Control to Null Controllable have been derived. These conditions are given with respect to Stability of Free Semi linear Base System and the Controllability of Semi linear Control Base System with the Assumption that the perturbation f satisfies some Smoothness and Growth Conditions. Computable Criteria for all these are reported. These results extended known results in the literature.

REFERENCES

[1] Oraekie,P.A.(2018);Euclidean Null Controllability of Nonlinear Infinite Neutral Systems with Multiple Delays in Control, Journal of the Nigerian Association of Mathematical Physics, Vol. 44 (Jan Issue),pp1-8.

[2] Brayton, R. (1976); Nonlinear Oscillations in a Distributed Network.Quart.Appl.Math. Pp 239-301.

[3] Ekgoltz, L.E.(1964); Qualitative Methods in Mathematical Analysis,Trans. Math. Mono, 12, American Math.Soc.

[4] Driver, R.D.(1963); A Functional Differential System of Neutral type in a two-body Problem of classical Electrodynamics in Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York.

[5] La Salle,J.P.(1959);The time Optimal Control Problem in Theory of Nonlinear Oscillations, Princeton Univ. Press, Princeton .N.J. Vol..5. ,pp.1-24 .

[6] Banks, H.T., M.Q..Jacobs and C.E.Langenhop (1975); Characterization of Controlled State in $W_2^{(1)}$ of Linear Hereditary Systems, SIAM J.Control Optimization 13, pp 611-649.

[7] Chukwu, E.N. (1997); Function Space Null Controllability of Delay Systems with Limited Power, J. of Mathematical Analysis and Applications, 124, pp 293-304.

[8] Oraekie, P.A. (2016); Relative Controllability of Nonlinear Infinite Space of Neutral Differential Systems with Distributive Delays in the Control, Journal of the Nigerian Association of Mathematical Physics, Vol 36, No.1, Pp 9-14.

[9] K.Naito (1992); On Controllability for a Nonlinear Volterra Equation, Nonlinear Analysis, Theory, Methods and Applications, 18, pp 99-108.

[10] ORAEKIE, P.A., (2017): Null Controllability of Nonlinear Infinite Space of Neutral Differential Systems with Distributed Delays in the Control, Journal of the Nigerian Association of MathematicalPhysics Vol.41, pp11-20.

[11] Dauer,J.P. and Balasubramaniam(1997) ; Null Controllability of Semi linear Integrodifferential Systems in Banach Spaces, Appl.Math.Lett. Vol.10, pp.117- 123.

[12] Rankin,S.M (1982),Existence and Asymptotic Behavior of a Functional Differential Equations in Banach Spaces, Journal of Mathematical Analysis and Applications, 88, pp531-542.

[13] Friedman,A.(1969); Partial Differential Equations ,Holt,Rinchart and Winston, New York.

[14] Pazy,A. (1983) ; Semi groups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlay, New York.

[15] Fitzgibbon,W.E.(1990) ;Asymptotic Behaviour of Solutions to a Class of Volterra Integrodifferential Equations, Journal ofMathematical Analysis and Applications, 146, pp241-253.
[16] Klamka,J. (1978) ;Relative Controllability of Nonlinear Systems with Distributed Delays in Control, International Journal of Control, 28, pp307-312.
[17] ORAEKIE, P.A., (2013):The Relative Controllability of NeutralVolterra Integrodifferential Systems with Zero in the Interior ofthe Reachable Set; African Journal of Sciences, Vol 14 No1, Page 3271-3282.
[18] Kantorovich,L.V.andAkilov,G.P.(1982) ; Functional Analysis.Pergaman Press Oxford.
[19] Onwuatu,J.U.(1993) ; Null Controllability of Nonlinear Infinite Neutral System, KYBERNETIKA,Volume 29,Number 4,pp 325-336.