International Journal of New Technology and Research (IJNTR)
ISSN: 2454-4116, Volume-5, Issue-3, March 2019 Pages 49-60

Existence and Uniqueness of Optimal Cotrol of Impulsive Quasi-Linear Fractional Mixed Volterra–Fredholm- Type Integro - Differential Euations in Banach Spaces with Multiple Delays in the Control

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Abstract— **The Impulsive Quasi - Linear Fractional Mixed Volterra - Fredholm – Type Integro – Differential Equations in Banach Spaces with Multiple Delays in the Control is presented for controllability analysis. Necessary and Sufficient Conditions for the Existence of an Optimal Control for the System are established. The mild solution of the system is established using the variation of constant formula. The set functions upon which our study hinged were extracted. Necessary and sufficient conditions for the establishment of the uniqueness of the system were derived. Use was made of some controllability standards to establish results. The establishment of the uniqueness of the optimal control provided a new approach for the proof of the existence of an optimal control of any dynamical control system. The main result is built on the maximization of a set function; a technique drawn from the calculus of variation.**

Index Terms— **Multiple Delays,Optimal Control,Relative Controllability, Impulsive Quasi – Linear, Fractional. Set Functions, Uniqueness.**

I. Introduction

Highlight The pioneering work of Vito Volterra on the Integration of the differential equations of dynamics and partial differential dynamical systems published in 1884 gave vent to the conception of integral equation of volterra type **Robertson [1].**It is equally observed in**Balachandran [2]**that the mixed initial boundary hyperbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed network**Balachandran [3]**which study has been extended to compartmental models governed by neutral Volterra integro-differential equations. Compartmental models have been found in **Burton [4]** to have numerous applications in Applied Mathematics; these models are used to vividly describe the evolutions of systems, in theoretical epidemiology, physiology, population dynamics chemical reaction kinetics and the analysis of ecosystems **Gyori [5] .**

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Most of these models can be divided into separate compartments. A paradigm for such systems can be seen as one in which compartments are connected by pipes through which materials pass in definite time. An example of compartmental model is given in **Gyori [5]**as the radio cardiogram where the two compartments correspond to the left and right ventricles of the heart and the pipe between these compartments represent the pulmonary and systemic circulations. Other applications of Volterra Integro-differential equation arise in tracer kinetics in the modeling of uptake of potassium by red blood cells as well as in modeling the kinetic of lead in a body**([4], [5]).** The wide application of Volterra integro-differential equations in Bio-Mathematics and economic models underscores the immense interest the study has generated. Literature on the relative controllability of Volterra equations is still scanty. However, sufficient conditions for the relative controllability of Non-linear neutral Volterra Integro-differential equations have been provided in **Balachandran [2].** However, the systems with delays in the state, investigation into their relative controllability are still attracting attention and interest. The controllability and approximate controllability of delay Volterra systems were investigated by using fixed point theorem **Oraekie [6] .** Thecontrollability and Local null controllability of Nonlinear Integrodifferential Systems and Functional Differential systems in Banach spaces were studied and it was shown that the controllability problem in Banach spaces can be converted into one of a fixed point problem for a single-valued mapping **[7].**.Balachandran and co-workers studied the controllability of Sobolev-ype Partial Functional Differential Systems in Banach Spaces **[8].** While **Oraekie [6]** studied the Retarded Functional Differential Systems of Sobolev-Type in Banach Spaces and established that once a system of the Sobolev – type is controllable with a single delay in the control of the system, then it is also controllable with either multiple delays or distributed delays or both multiple and distributed delays in the control .However, necessary and sufficient conditions for the target set of Nonlinear Infinite spaces of Functional Differential systems with Distributed Delays in the control to be on the boundary of the Attainable set of the system have been provided in **Oraekie [9]** .It was made clear that whenever an optimal control is in use to steer the system of interest from the initial point to the target (**desired point**) , then the target set must be on the boundary of the attainable set of the system **Oraekie [10] .**Optimality conditions for the relative controllability of neutral Volterra Integro-differential equation has been provided by **Oraekie [11] ;** though there are studies in the optimal controllability of ordinary and functional differential systems. From (**[12], [13], [14]),** we gain clarity of meaning and understanding of the conceptual frame work of optimal controllability.

Many processes studied Applied Sciences are represented by differential Equations. However, the situation is quite different in many physical phenomena that have sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flow, population dynamics, theoretical physics, radio physics, pharmacokinetics, mathematical economy, chemical technology, biotechnology and medicine etc. Adequate mathematical models of such processes are systems of differential equation with impulses.Thetheory of impulsive differential and integro-differential equations is a new and important branch of differential equations **([15], [16]).**

Fractional differential equationshave recently proved to be valuable tools in the modeling of many phenomena in various fields of sciences and engineering (**Banila[17], [18]).** There has been significant development in fractional differential equations in recent years**([19], [20], and [21]) Recently, Balachandran [22]** studied the existence results for Impulsive fractional differential and Integro-differential equations in Banach spaces using standard fixed point theorems.

Controllability is the most important qualitative behavior of any dynamical system. It is well known that the issue of controllability plays an important role in control theory and engineering**([23], [24] and [25])** because they have close connections to structural decomposition, quadratic optimal control, observer design etc. The literature related to controllability of Impulsive fractional Integro-differential equations and controllability of Impulsive Quasi-linear Integro-differential equations is limited, to our knowledge, to the recent works **([17], [26]).**The study of controllability of Impulsive Quasi-linear fractional mixed Volterra-Fredholm-type integro-differential equations is presented in **Kavitha [27].**

In this paper, therefore, we shall considerImpulsive Quasi −linear fractional mixed Volterra − Fredholm − type Integro − differential equations in Banach spaces with Multiple delays in the control of the form:

$$\boldsymbol{D^q x}(t) = A(t,x)x(t) + \sum_{j=0}^{m} B_j u\left(t-h_j\right) + \quad f\left(t, x(t), \int_0^t g(t,s,x(s))\,ds, \int_0^b k(t,s,x(s))ds\right) (1.1)$$

$t \in J = [0,b], t \neq t_k, k = 1,2,3,\dots,m$

$$\Delta_x|_{t=t_k} = I_k\left(x(t_k{}^{-1})\right), k = 1,2,3,\dots,m \qquad (1.2)$$

$$x(0) = x_0 \qquad (1.3)$$

with the main objective, of investigating the existence of an optimal control of the systems (1.1)

1.1. *Description of System*

Consider system (1.1) – (1.3) – the impulsive quasi-linear fractional mixed Volterra-Fredholm-type integro-differential equations in Banach spaces with multiple delays in the control given above. That is,

$$\boldsymbol{D^q x}(t) = A(t,x)x(t) + \sum_{j=0}^{m} B_j u\left(t-h_j\right) + \quad f\left(t, x(t), \int_0^t g(t,s,x(s))\,ds, \int_0^b k(t,s,x(s))ds\right) (1.1)$$

$t \in J = [0,b], t \neq t_k, k = 1,2,3,\dots,m$

$$\Delta_x|_{t=t_k} = I_k\left(x(t_k{}^{-1})\right), k = 1,2,3,\dots,m \qquad (1.2)$$

$$x(0) = x_0 \qquad (1.3)$$

Here, the state variable $x(.)$ *takes values in the Banach space* X *and* $u(.)$ *is a control function is an admissible square integrable* $m-$ *dimensional vector function, with* U *as a Banach space. i. e.* $u \in L_2(J, U)$ *. Here,* $0 < q < 1$, *and* $A(t,x)$ *is a bounded linear operator on a Banach space* X *. Further more,* $f : JxXxXxX \to X, g : \Omega xX \to X$, $k : \Omega xX \to X, I_k : X \to X, \Delta_x I_{t=t_k} = x((t_k{}^+) - x(t_k{}^-)$, *for all* $k = 1,2,\dots,m$;

$0 < t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = t_1;\ \Omega = \{(t,s), 0 \le s \le t_1\}$.

Thus the control space will be the Lebesgue space of square integrable functions.

The constraint control set U *is the closed and bounded subset of* L_2.

Let $h > 0$ *, be given. For a function* $u: [-h, t_1] \to X$ *and* $t \in [t_0, t_1]$,

we use the ssymbol $\boldsymbol{u_t}$ *to denote the function defined on the delay interval* $[-h, 0]$

by $\boldsymbol{u_t(s) = u(t+s)}$, ***for*** $\boldsymbol{s \in [-h, 0]}$.

Definition 2.1. (***complete state***)

The complete state for system(1.1)*is given by the set*

$$\boldsymbol{z(t) = \{x, u_t\}}$$

Definition 2.2.

The system(1.1) *is said to be relatively controllable on the interval* $[t_0, t_1]$*if for every initial complete state* $z(0)$ *and* $x_1 \in X$, *there exists a control function* $u(t)$

$defined\ on\ [t_0\ ,t_1]\ such\ that\ the\ solution\ x(.)\ of\ the\ system\ (1.1) satisfies\ \ x(t_1) = x_1$

2.3. Basic Definitions of Fractional Calculus

$Let\ X\ \ be\ a\ Banach\ space\ \ and\ R^+ = [0,\infty).\ Suppose\ \ f \in L_1(R^+)\ .\ Let\ \ C(J\ ,X\ )\ be\ the$ $Banach\ space\ of\ continuous\ functions\ x(t)\ with\ x(t) \in X\ for\ t \in J = [t_0\ ,t_1]\ \ \ and$

$$\|x\|_{C(J\ ,X\ )} = \max_{t\in J}\|x(t)\|\ .$$

$Let\ B(X) denote\ the\ Banach\ space\ of\ bounded\ linear\ operators\ from\ X\ into\ X$ $with\ the\ norm\qquad \|A\|_{B(X)} = sup\{\|A(y)\|:\ \|y\| = 1\ \}$

$Also\ consider\ the\ Banach\ space\ \ PC(J\ ,X\ )\ \ = \{x : J \to X : x \in C\{[t_k\ ,t_{k+1}],X\ )\}\ ,$ $k = 0,1,2,\dots\ ,m\ and\ there\ exists\ x(t_k{}^-)\ and\ x((t_k{}^+)\ with\ x((t_k{}^-) = x(t_k)$ $with\ the\ norm\qquad \|x\|_{PC} = \operatorname{Sup}_{t\in J}\|x(t)\|\ .$

$\boldsymbol{Definition}\ \mathbf{2.3.1}$

$The\ Riemann-Liouville\ fractional\ integral\ operator\ of\ order\ \alpha > 0, of\ function$ $f \in L_1(R^+)\ \ is\ defined\ as$

$$\int_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds$$

$where\ \Gamma(.)\ \ is\ the\ Euler\ Gamma\ function.$

Definition 2.3.2

$The\ Riemann-Liouville\ fractional\ derivative\ of\ order\ \alpha > 0, n-1 < \alpha < n\ ,n \in N$

$$is\ defined\ as\ \ :\ D_{0^+}{}^{\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}\,\mathrm{f}(s)\mathrm{ds}\ ,$$

$Where\ the\ function\ f(t) has\ absolutely\ continuous\ derivatives\ up\ to\ order\ (n-1).$

Definition 2.3.3

$The\ Caputo\ fractional\ derivative\ of\ order\ \ \ \alpha > 0, n-1 < \alpha < n\ ,n \in N\ is\ defined\ as:$

$$D_{0^+}{}^{\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\,\mathrm{f}^n(s)\mathrm{ds}\ ,$$

$Where\ the\ function\ f(t)\ has\ absolutely\ continuous\ derivatives\ up\ to\ order\ (n-1).$

$If\ n = 1\ ,\qquad then\ ,\qquad n-1 < \ \alpha < n = 0 < \alpha < 1\ , implies\ that$

$$D_{0^+}{}^{\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{1-\alpha-1}\,\mathrm{f}^1(s)\mathrm{ds}\ = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,\mathrm{f}^1(s)\mathrm{ds}$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-s)^{\alpha}}\mathrm{f}^1(s)\mathrm{ds}\ = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{f}^1(s)}{(t-s)^{\alpha}}\mathrm{ds}$$

$$Where\ ,\qquad \mathrm{f}^1(s) = Df(s) = \frac{df(s)}{ds}\ \ and\ f\ is\ an\ \ abstract\ \ function\ with\ values\ in\ X.$$

2.4. *Variation of Constant Formula*

From the works of**[21]**and **[28]** **,**we have theMild solutionof systems (1.1) – (1.3) as the following integral equation :

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s\ ,x)x(s)ds\ + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s\ ,x)x(s)ds\ +$$
$$\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\left[\sum_{j=0}^{m}B_j(s)u(t-h_j)\right]ds$$
$$+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f\left(s\ ,x(s)\ ,\int_0^s g(s\ ,\tau\ ,x(\tau))d\tau\ ,\int_0^b k(s\ ,\tau\ ,x(\tau))d\tau\right)ds$$
$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\sum_{j=0}^{m}B_j\ (s)u(s-h_j) + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}f\left(s\ ,x(s)\ ,\int_0^s g(s\ ,\tau\ ,x(\tau))d\tau\ ,\int_0^b k(s\ ,\tau\ ,x(\tau))d\tau\right)ds$$
$$+\sum_{0<t_k<t} I_k(x\ (t_k{}^-)) \qquad (2.3.1)$$

$One\ may\ assume\ without\ loss\ of\ generality\ that\ ,$

$$h_m > h_{m-1} > h_{m-2} > \ldots > h_1 > h_0 = 0$$

$$The\ initial\ control\ \ u_0(t)\ \ is\ given\ on\ \left[t_0 - h_m\ ,t_0\right]\ ,t_0 = 0$$

$The\ solution\ of\ the\ system (1.1)\ for\ \ \ t > t_0 + h_m\ is\ given\ by$

$$x(t,x_0,u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s,x)x(s)ds$$
$$+\sum_{j=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \sum_{j=0}^{m} B_j(s)u(t-h_j)\,ds$$
$$+\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds$$
$$+\frac{1}{\Gamma(q)} \sum_{j=0}^{m} B_j(s)u(s-h_j) \int_{t_k}^{t} (t-s)^{q-1} ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds$$
$$+\sum_{0<t_k<t} I_k\left(x(t_k^-)\right) \qquad (2.3.2)$$

$$x(t,x_0,u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s,x)x(s)ds$$
$$+\sum_{j=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_0}^{t_1} (t_k - s + h_j)^{q-1} \sum_{j=0}^{m} B_j(s+h_j)u(t-h_j+h_j)\,ds, \boldsymbol{for\ k=1}$$
$$+\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds$$
$$+\sum_{j=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_j}}^{t-h_j} (t-s+h_j)^{q-1} B_j(s+h_j)u(t-h_j+h_j)ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds$$
$$+\sum_{0<t_k<t} I_k\left(x(t_k^-)\right) \qquad (2.3.3)$$

$$\Rightarrow, x(t,x_0,u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s,x)x(s)ds$$
$$+\sum_{j=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{0-h_j}}^{t_0} (t_0 - s + h_j)^{q-1} B_j(s+h_j)u_0(s-h_j+h_j)ds, \boldsymbol{for\ k=1}$$
$$+\sum_{j=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{1-h_j+1}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s+h_j)u(s-h_j+h_j)ds$$
$$+\frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds$$
$$+\sum_{j=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_j}}^{t-h_j} (t-s+h_j)^{q-1} B_j(s+h_j)u(t-h_j+h_j)ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds$$

$$+\sum_{0<t_k<t} I_k\left(x\left(t_k^{-}\right)\right) \qquad (2.3.4)$$

$$\Rightarrow, x(t, x_0, u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, x)x(s)ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s, x)x(s)ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{0-h_j}}^{t_0} \left(t_0 - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u_0\left(s - h_j + h_j\right) ds, \boldsymbol{for\ k = 1}$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{1-h_j+1}}^{t_{1-h_j}} \left(t_1 - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u\left(s - h_j + h_j\right) ds$$
$$+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau, \int_0^b k(s, \tau, x(\tau))d\tau\right) ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t-h_j} \left(t - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u\left(s - h_j + h_j\right) ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f\left(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau, \int_0^b k(s, \tau, x(\tau))d\tau\right) ds$$
$$+ \sum_{0<t_k<t} I_k\left(x\left(t_k^{-}\right)\right) \ (2.3.5)$$

$$Put\ \boldsymbol{G(s)} = f\left(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau, \int_0^b k(s, \tau, x(\tau))d\tau\right),$$

$system(2.3.5)\ becomes$

$$x(t, x_0, u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, x)x(s)ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s, x)x(s)ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{0-h_j}}^{t_0} \left(t_0 - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u_0(s) ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{1-h_j+1}}^{t_{1-h_j}} \left(t_1 - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u(s) ds$$
$$+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \boldsymbol{G(s)}\ ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t-h_j} \left(t - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u(s) ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \boldsymbol{G(s)}\ ds + \sum_{0<t_k<t} I_k\left(x\left(t_k^{-}\right)\right) \ (2.3.6)$$

$For\ brevity, let$

$$\boldsymbol{\mu(t)} = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, x)x(s)ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s, x)x(s)ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{0-h_j}}^{t_0} \left(t_0 - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u_0(s) ds \qquad (2.3.7)$$
$$\boldsymbol{\beta(t)} = \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \boldsymbol{G(s)}\ ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \boldsymbol{G(s)}\ ds$$

$$+\sum_{0<t_k<t} I_k\big(x\,(t_k{}^{-})\big) \qquad (2.3.8)$$

$$\mathbf{Z}(\boldsymbol{t},\boldsymbol{s})\mathrm{u}(s)\mathrm{ds} = \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_j+1}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds$$
$$+\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\left(t-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds \qquad (2.3.9)$$

Substituting equations $(2.3.7)$ ***and*** $(2.3.8)$ ***in*** $(2.3.6)$ ***, we have the mild solution:***

$$\boldsymbol{x}(\boldsymbol{t}\ ,\boldsymbol{x_0}\ ,\boldsymbol{u}\ ) = \boldsymbol{\mu}(\boldsymbol{t})+\boldsymbol{\beta}(\boldsymbol{t})+\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{q})}\sum_{\mathbf{0}<\boldsymbol{t_1}<\boldsymbol{t}}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t_{1-h_j}}}\left(\boldsymbol{t_1}-\boldsymbol{s}+\boldsymbol{h_j}\right)^{\boldsymbol{q-1}}\boldsymbol{B_j}\left(\boldsymbol{s}+\boldsymbol{h_j}\right)\boldsymbol{u}(\boldsymbol{s})\boldsymbol{ds}$$
$$+\sum_{]=\mathbf{0}}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{q})}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t-h_j}}\left(\boldsymbol{t}-\boldsymbol{s}+\boldsymbol{h_j}\right)^{\boldsymbol{q-1}}\boldsymbol{B_j}\left(\boldsymbol{s}+\boldsymbol{h_j}\right)\boldsymbol{u}(\boldsymbol{s})\boldsymbol{ds} \qquad (\mathbf{2.3.10})$$

2.4 . Basic Set Function and Properties

We shall define the set functions upon which our study hinges

Definition 2.4.1.(reachable set)

The reachable set $R(t_1,t_0)$ *of the systems* $(2.1)-(2..3)$ *is given as*

$$\boldsymbol{R}(t_1,t_0)=\left\{\begin{array}{l}\displaystyle\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{q})}\sum_{\mathbf{0}<\boldsymbol{t_1}<t}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t_{1-h_j}}}\left(\boldsymbol{t_1}-\boldsymbol{s}+\boldsymbol{h_j}\right)^{\boldsymbol{q-1}}\boldsymbol{B_j}\left(\boldsymbol{s}+\boldsymbol{h_j}\right)\boldsymbol{u}(\boldsymbol{s})\boldsymbol{ds}\\ \displaystyle+\sum_{]=\mathbf{0}}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{q})}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t-h_j}}\left(\boldsymbol{t}-\boldsymbol{s}+\boldsymbol{h_j}\right)^{\boldsymbol{q-1}}\boldsymbol{B_j}\left(\boldsymbol{s}+\boldsymbol{h_j}\right)\boldsymbol{u}(\boldsymbol{s})\boldsymbol{ds}\ :\\ \left|\boldsymbol{u_j}\right|\le\mathbf{1}\ \boldsymbol{for\ every\ j and}\ \boldsymbol{u_j}\in\boldsymbol{U}\subseteq\boldsymbol{L_2}(\boldsymbol{J}\ ,\boldsymbol{X^m});\boldsymbol{j}=\mathbf{1}\ ,\mathbf{2}\ ,\dots,\boldsymbol{m}\end{array}\right\}$$

Definition 2.4.2. (attainable set)

The attainable set $A(t_1,t_0)$ *of the systems* $(2.1)-(2..3)$ *is given as*

$$A(t_1,t_0)=\left\{\boldsymbol{x}(\boldsymbol{t}\ ,\boldsymbol{x_0}\ ,\boldsymbol{u}\ ):\boldsymbol{u}\in\boldsymbol{U}\right\},$$

$$\boldsymbol{where}\ \boldsymbol{U}=\left\{\boldsymbol{u}\in\boldsymbol{L_2}([t_0\ ,t_1]\ ,\boldsymbol{X^m}):\left|\boldsymbol{u_j}\right|\le\mathbf{1}\ ;\ \boldsymbol{j}=\mathbf{1}\ ,\mathbf{2}\ ,\dots,\boldsymbol{m}\right\}$$

Definition 2.4.3. (target set)

The target set $G(t_1,t_0)$ *of the systems* $(2.1)-(2..3)$ *is given as*

$$\boldsymbol{G}(t_1,t_0)=\{\boldsymbol{x}(\boldsymbol{t}\ ,\boldsymbol{x_0}\ ,\boldsymbol{u}):\boldsymbol{t}\ge\boldsymbol{\tau}>\boldsymbol{t_0}=\mathbf{0}\ ,\boldsymbol{for\ fixed}\ \boldsymbol{\tau}\ \boldsymbol{and}\ \boldsymbol{u}\in\boldsymbol{U}\}\ .$$

Definition 2.4.4.(controllability grammian)

The controllability grammian $W(t_1,t_0)$ *of the systems* $(2.1)-(2..3)$ *is given as*

$\boldsymbol{W}(t_1,t_0)=\ \mathbf{Z}(\boldsymbol{t},\boldsymbol{s})\mathbf{Z}^{\boldsymbol{T}}(\boldsymbol{t},\boldsymbol{s})$ *, where T denotes matrix transpose.*

2.5. Relationship between the set functions

We shall first establish the relationship between the attainable set and the reachable set to enable us see that once a property has been proved for one set, and then it is applicable to the other.

From the equation $(2.3.6)$*, we have the attainable set* $A(t_1,t_0)$ *as:*

$$\boldsymbol{A}(t_1,t_0)=[\ \boldsymbol{\eta}(\boldsymbol{t})+\boldsymbol{R}(t_1,t_0)]\ ,$$

for $u\in U\ ,t\in[t_0\ ,t_1]$ *, where* $\boldsymbol{\eta}(\boldsymbol{t})=\boldsymbol{\mu}(\boldsymbol{t})\ +\boldsymbol{\beta}(\boldsymbol{t})$.

This means that the attainable set is the translation of the reachable set through the origin $\boldsymbol{\eta}\in\boldsymbol{X^n}$*. Using the attainable set, therefore, it is easy to show that the s et functions possess the prperties of conexity , closedness , boundedness, , and compactness. Also, the set functions are continuous on* $[0\ ,\infty)$ *to the metric space of compact subsets of* X^n *.*

$[12]$ ***and*** $[29]$ *gave the impetus for adoption of the proofs of these properties for systms*$(2.1)-(2.3)$ *or systems* $(1.1)-(1.3)$.

Definition 2.5.1(relative controllability)

System(2.1) *is relatively controllable on the interval* $[t_0\ ,t_1]$ *if*

$$A(t_1,t_0)\cap G(t_1,t_0)\neq\emptyset\ ;t_1>t_0$$

Definition 2.5.2. (properness)

System(2.1) *is proper in X on* $[t_0\ \ ,t_1]$ *if span* $R(t_1,t_0)=X^n$

$$i.e.\quad C^T\sum_{]=\mathbf{0}}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{q})}\sum_{\mathbf{0}<\boldsymbol{t_1}<t}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t_{1-h_j}}}\left(\boldsymbol{t_1}-\boldsymbol{s}+\boldsymbol{h_j}\right)^{\boldsymbol{q-1}}\boldsymbol{B_j}\left(\boldsymbol{s}+\boldsymbol{h_j}\right)\boldsymbol{u}(s)ds$$

$$+C^T \sum_{j=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t-h_j} (t-s+h_j)^{q-1} B_j(s+h_j)u(s)ds = 0 \ a.e \ , \Rightarrow C = 0, C \in X^n .$$

3. Main Work

The optimal control problem can best be understood in the context of a game of pursuit ([30], [31]). The emphasis here is the search for a control energy function that can steer the state of the system of interest to the target set which can be a moving point function or a compact set function in minimum time. In other words, the optimal control problem is stated as follows: If

$$t^* = infimum\{t : A(t_1, t_0) \cap G(t_1, t_0) \neq \emptyset \ for \ t \in [t_0, t_1]\}, t_1 > t_0$$

That is, if t^* is the infimum of all the times such that the system is relatively controllable, Is there an admissible control u^* such that the solution of the system with this admissible control be steered into target? The theorem that follows answ in part the questioners.

Theorem 3.1. **(*Existence Condition*)**

Consider the systems (1.1) − (1.3) as a differential game of pursuit:

$$D^q x(t) = A(t, x)x(t) + \sum_{j=0}^{m} B_j u(t-h_j) + f\left(t, x(t), \int_0^t g(t, s, x(s))\, ds, \int_0^b k(t, s, x(s))ds\right) \quad (1.1)$$

$t \in J = [0, b], t \neq t_k, k = 1,2,3, \dots, m$

$$\Delta_x|_{t=t_k} = I_k(x(t_k^{-1})), k = 1,2,3, \dots, m \quad (1.2)$$

$$x(0) = x_0 \quad (1.3)$$

with its standing assumptions. Suppose $A(t_1, t_0)$ and $G(t_1, t_0)$ are compact set functions, then there exists an admissible control such that the state of the weapon for the pursuit of the target satisfies systems (1.1) − (1.3) if and only if

$$A(t_1, t_0) \cap G(t_1, t_0) \neq \emptyset \ for \ t \in [t_0, t_1].$$

Proof

Suppose that the state $z(t)$ of the weapon for pursuit of the target satisfies systems (1.1) − (1.3) on the time interval $[t_0, t_1]$, then $z(t) \in G(t_1, t_0)$ for $t \in [t_0, t_1]$. We need to show that there exists $x(t, u) \in A(t_1, t_0)$ for $t \in [t_0, t_1]$ such that

$$z(t) = x(t, u), for\ some\ u \in U.$$

Let $\{u^n\}$ be a sequence of points in U. since the constraint control set is compact, then the sequence $\{u^n\}$ has a limit u as n tends to infinity

$$.i.e. \lim_{n \to \infty} u^n = u$$

Now $x(t, x_0, u) \in A(t_1, t_0)$, for $t \in [t_0, t_1]$ and from system(2.3.6) we have

$$\begin{aligned} x(t, x_0, u^n) = {} & x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, x)x(s)ds \\ & + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s, x)x(s)ds \\ & + \sum_{j=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{0-h_j}}^{t_0} (t_0 - s + h_j)^{q-1} B_j(s+h_j)u_0{}^n(s)ds \\ & + \sum_{j=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{1-h_j+1}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s+h_j)u^n(s)ds \\ & + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} G(s)\, ds \\ & + \sum_{j=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t-h_j} (t-s+h_j)^{q-1} B_j(s+h_j)u^n(s)ds \\ & + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} G(s)\, ds + \sum_{0<t_k<t} I_k(x(t_k^-)) . \end{aligned}$$

Taking limits on both sides asnn tends to infinity, we have

$$\lim_{n\to\infty} x(t\ ,x_0\ ,u^{\ n}) \ = \ x_0 + \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s\ ,x)x(s)ds$$

$$+\ \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s\ ,x)x(s)ds$$

$$+\ \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{0-h_j}}^{t_0}\left(t_0-s+h_j\right)^{q-1}B_j\left(s+h_j\right)\lim_{n\to\infty}u_0{}^n(s)ds$$

$$+\ \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_j+1}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)\lim_{n\to\infty}u^n(s)ds$$

$$+\ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\,\boldsymbol{G(s)}\,ds$$

$$+\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\left(t-s+h_j\right)^{q-1}B_j\left(s+h_j\right)\lim_{n\to\infty}u^n(s)ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\boldsymbol{G(s)}\,ds+\sum_{0<t_k<t}I_k\left(x\left(t_k{}^{-}\right)\right)$$

$$=\ x(t\ ,x_0\ ,u\ )\ =\ x_0+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s\ ,x)x(s)ds$$

$$+\ \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s\ ,x)x(s)ds$$

$$+\ \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{0-h_j}}^{t_0}\left(t_0-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u_0{}^n(s)ds$$

$$+\ \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_j+1}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u^n(s)ds$$

$$+\ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\,\boldsymbol{G(s)}\,ds$$

$$+\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\left(t-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u^n(s)ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\boldsymbol{G(s)}\,ds+\sum_{0<t_k<t}I_k\left(x\left(t_k{}^{-}\right)\right)\ =\ x(t\ ,x_0\ ,u\ )\ \in\ A(t_1,t_0)\ ,$$

Since $A(t_1,t_0)$ *is compact.*

Thus , there exists a control $u\in U$ *such that* $x(t\ ,x_0\ ,u\ )=z(t)$ *for* $t>t_0$ *and* $t\in[t_0\ ,t_1]$.

Since $z(t)\in G(t_1,t_0)$ *and also in* $A(t_1,t_0)$*, it follows that* $A(t_1,t_0)\cap G(t_1,t_0)\neq\emptyset$ *for* $t\in[t_0\ ,t_1]$.

Conversely, suppose that the intersection condition holds .

i.e. $A(t_1,t_0)\cap G(t_1,t_0)\neq\emptyset$ *for* $t\in[t_0\ ,t_1]$,

Then there exists $z(t)\in A(t_1,t_0)$ *suchthat* $z(t)\in G(t_1,t_0)$*. This implies that* $z(t)=x(t\ ,x_0\ ,u\ )$ *and hence establishes that the state of the weapon of pursuit of the target satisfies systems* (1.1) − (1.3). This completes the proof.

***REMARK* 3.4**

The above stated and prove theorem3.4 in other words states that ,in any game of pursuit described byImpulsive Quasi-Linear Fractional Mixed Volterra - Fredholm Type Integro – Differential Equations in Banach Spaces with Multiple Delays in the Control, it is always possible to obtain a control energy function to steer the systems’ state to the target in finite time .However,a pyrrhic victory is desirable and is therefore,the insistence for the search of an optimal control.The next theorem is therefore,a consequence of this understanding and provides sufficient conditions for the existence of a control energy function that is capable of steering the state of the systems (1.1)– (1.3)via-a-via systems (2.1) – (2.3) to target set in minimum time.

3.5. Sufficient Conditions for the Existence of an Optimal Control

The theorem below informs us that relative controllability of a system guarantees the existence of its optimal control **[32].**

Theorem 3.2

Consider the systems(1.1)– (1.3)via – a – via systems (2.1)– (2.3)

$$\boldsymbol{D^q x}(t) = A(t,x)x(t) + \sum_{j=0}^{m} B_j u\left(t-h_j\right) + \quad f\left(t, x(t), \int_0^t g(t,s,x(s))\,ds, \int_0^b k(t,s,x(s))ds\right) \quad (1.1)$$

$t \in J = [0,b], t \neq t_k, k = 1,2,3,\dots,m$

$\Delta_x|_{t=t_k} = I_k\left(x(t_k{}^{-1})\right), k = 1,2,3,\dots,m \qquad (1.2)$

$x(0) = x_0 \qquad (1.3)$

With its basic assumptions. Suppose that system (1.1) *is relatively controllable on the finite interval* $[t_0, t_1]$*, then there exists an optimal control. By the controllability of system*(1.1), *the intersection condition holds.*

$$That\ is,.\ A(t_1,t_0) \cap G(t_1,t_0) \neq \emptyset \ for\ t \in [t_0,t_1].$$

Hence, $x(t,x_0,u) \in A(t_1,t_0)$. *Also*, $x(t,x_0,u) \in G(t_1,t_0)$. *So, put* $z(t) = x(t,x_0,u)$.
Recall that the attainable set $A(t_1,t_0)$ *is a translation of the reachable set* $R(t_1,t_0)$ *through the origin* η,

$$i.e.\ A(t_1,t_0) = \eta + R(t_1,t_0),$$

where η *is given as* :

$$\eta = \mu(t) + \beta(t),$$

then

$$\begin{aligned} A(t_1,t_0) = x_0 &+ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s,x)x(s)ds \\ &+ \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s,x)x(s)ds \\ &+ \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_0-h_j}^{t_0}\left(t_0-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u_0(s)ds \\ &+ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\boldsymbol{G(s)}\,ds + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\boldsymbol{G(s)}\,ds \\ &+ \sum_{0<t_k<t} I_k\left(x\left(t_k{}^{-}\right)\right) + R(t_1,t_0). \end{aligned}$$

It follows that $z(t) \in R(t_1,t_0)$ for $t \in [t_1,t_0], t_1 > t_0$ and can be defined as:

$$z(t) = \left\{\begin{array}{l} \displaystyle\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_j+1}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds \\ \displaystyle\quad+\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\left(t-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds \ : \\ \left|u_j\right| \leq 1, for\ every\ j\ and\ u_j \in U \subseteq L_2(J,X^m); j = 1,2,\dots,m \end{array}\right\}$$

Let $\boldsymbol{t}^* = infimum\{t : z(t) \in R(t_1,t_0)$, for $t \in [t_1,t_0]\}$. *Now* $0 \leq t_n \leq t_1$ *and there is a sequence of times* $\{\boldsymbol{t_n}\}$ *and the corresponding sequence of controls* u^n : $\{\boldsymbol{u^n}\} \subset U$ *with the sequence* $\{\boldsymbol{t_n}\}$ *convrging to* $\boldsymbol{t}^*$ *, the minimum time.*
Let, $z(t_n) = y(t_n, u^n) \in R(t_1,t_0)$. *Also*

$$\begin{aligned} |z(\boldsymbol{t}^*) - y(\boldsymbol{t}^*,\boldsymbol{u^n})| &\leq |z(\boldsymbol{t}^*) - z(t_n) + z(t_n) - y(\boldsymbol{t}^*,\boldsymbol{u^n})| \\ &\leq |z(\boldsymbol{t}^*) - z(t_n)| + |z(t_n) - y(\boldsymbol{t}^*,\boldsymbol{u^n})| \\ &\leq |z(\boldsymbol{t}^*) - z(t_n)| + |y(t_n,u^n) - y(\boldsymbol{t}^*,\boldsymbol{u^n})| \\ &\leq |z(\boldsymbol{t}^*) - z(t_n)| + \int_{t^*}^{t_n}\|y(s)\|\,ds. \end{aligned}$$

By the continuity of $z(t)$. *which follows the continuity of* $R(t_1,t_0)$ *as a continuous set function and the integrability of* $\|y(s)\|$, *it follows thaty* $(\boldsymbol{t}^*,\boldsymbol{u^n}) \to z(t^*)$ *as* $n \to \infty$, *where*,

$$z(\boldsymbol{t}^*) = y(t^*,u^*) \in R(t_1,t_0).$$

For some $u^* \in U$ *and by the definition of* $\boldsymbol{t}^*$; u^* ***is an optimal control.***

This establishes the existence of an optimal control for the Impulsive Quasi – linear Fractional Mixed Volterra – Fredholm – Type Integro – differential Equations in Banach spaces with Multiple Delays in the Control.

3.6 .Uniqueness of the Optimal Control.

Here, a new method of approach is derived for the proof of the existence of optimal control

***Theorem* 3.3**

Consider the systems(1.1)– (1.3)via – a – via systems (2.1)– (2.3)

$$\boldsymbol{D^q x}(t) = A(t\,,x)x(t) + \sum_{j=0}^{m} B_j u\big(t-h_j\big) + \quad f\left(t\,,x(t)\,,\int_0^t g(t\,,s\,,x(s))\,ds\,,\int_0^b k(t\,,s\,,x(s))ds\right)(1.1)$$

$t \in J = [0\,,b], t \neq t_k\,, k = 1,2,3,\ldots,m$

$$\Delta_x|_{t=t_k} = I_k\big(x(t_k{}^{-1})\big)\,, k = 1,2,3,\ldots,m \qquad (1.2)$$

$$x(0) = x_0 \qquad (1.3)$$

With its basic assumptions. Suppose that system (1.1) *is relatively controllable on the finite interval* $[t_0\,,t_1]$. *Suppose that* u^* *the optimal control of system*(1.1), *then it is unique.*

Proof

Let $\boldsymbol{u}^*$ *and* $\boldsymbol{v}^*$ *be optimal controls of system*(1.1), *then both* $\boldsymbol{u}^*$ *and* $\boldsymbol{v}^*$ *maximize*

$$\left[C^T\sum_{]=0}^{m}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\sum_{\boldsymbol{0<t_1<}t}\big(\boldsymbol{t_1-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\;+C^T\sum_{]=0}^{m}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\big(\boldsymbol{t-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]$$

For $t \in [t_0\,,t_1], t_1 > 0$, *over all admissible controls* $u \in U$, *and so we have the inequality with* u^* *as the optimal control*

$$\boldsymbol{C^T}\int_{\boldsymbol{t_{1-h_j+1}}}^{\boldsymbol{t_{1-h_j}}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\sum_{\boldsymbol{0<t_1<}t}\big(\boldsymbol{t_1-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u(s)}\,\mathbf{ds}$$
$$+\boldsymbol{C^T}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t-h_j}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\big(\boldsymbol{t-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u(s)ds}$$

$$\leq \quad \boldsymbol{C^T}\int_{\boldsymbol{t^*{}_{1-h_j+1}}}^{\boldsymbol{t^*{}_{1-h_j}}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\sum_{\boldsymbol{0<t_1<}t}\big(\boldsymbol{t^*{}_1-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u^*(s)ds}$$
$$+\boldsymbol{C^T}\int_{\boldsymbol{t^*{}_{1-h_{j+1}}}}^{\boldsymbol{t^*-h_j}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\big(\boldsymbol{t^*-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u^*(s)}\mathbf{ds} \qquad (3.3)$$

Also*, *using* $\boldsymbol{v^*}$, *asthe optimal control*, *we have the inequality

$$\boldsymbol{C^T}\int_{\boldsymbol{t_{1-h_j+1}}}^{\boldsymbol{t_{1-h_j}}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\sum_{\boldsymbol{0<t_1<}t}\big(\boldsymbol{t_1-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u(s)}\,\mathbf{ds}$$
$$+\boldsymbol{C^T}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t-h_j}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\big(\boldsymbol{t-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u(s)ds}$$

$$\leq \quad \boldsymbol{C^T}\int_{\boldsymbol{t^*{}_{1-h_j+1}}}^{\boldsymbol{t^*{}_{1-h_j}}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\sum_{\boldsymbol{0<t_1<}t}\big(\boldsymbol{t_1{}^*-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{v^*(s)ds}$$
$$+\quad\boldsymbol{C^T}\int_{\boldsymbol{t^*{}_{1-h_{j+1}}}}^{\boldsymbol{t^*-h_j}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\big(\boldsymbol{t^*-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{v^*(s)}\mathbf{ds} \qquad \mathbf{(3.4).}$$

Taking maximum of u *over* $[-1\,,1]$, *the range of definition of* u^*, v^* *in* (3.3) *and* (3.4) *respectively, we have the equations.*

$$\boldsymbol{C^T}\int_{\boldsymbol{t_{1-h_j+1}}}^{\boldsymbol{t_{1-h_j}}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\sum_{\boldsymbol{0<t_1<}t}\big(\boldsymbol{t_1-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u(s)}\,\mathbf{ds}$$
$$+\quad\boldsymbol{C^T}\int_{\boldsymbol{t_{1-h_{j+1}}}}^{\boldsymbol{t-h_j}}\left[\sum_{]=0}^{\boldsymbol{m}}\frac{\mathbf{1}}{\boldsymbol{\Gamma(q)}}\big(\boldsymbol{t-s+h_j}\big)^{\boldsymbol{q-1}}\boldsymbol{B_j\big(s+h_j\big)}\right]\boldsymbol{u(s)ds}$$

$$= \quad C^T \int_{t^*_{1-h_{j+1}}}^{t^*_{1-h_j}} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1{}^* - s + h_j)^{q-1} B_j(s+h_j) \right] u^*(s)ds$$

$$+ \; C^T \int_{t^*_{1-h_{j+1}}}^{t^*-h_j} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t^* - s + h_j)^{q-1} B_j(s+h_j) \right] u^*(s)\mathbf{ds} \qquad (3.5).$$

$for \; -1 \le s \le 1 \; and \; u, u^* \; \in U$

Also,

$$C^T \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1 - s + h_j)^{q-1} B_j(s+h_j) \right] u(s)\,\mathbf{ds}$$

$$+ \; C^T \int_{t_{1-h_{j+1}}}^{t-h_j} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t - s + h_j)^{q-1} B_j(s+h_j) \right] u(s)ds$$

$$= \quad C^T \int_{t^*_{1-h_{j+1}}}^{t^*_{1-h_j}} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1{}^* - s + h_j)^{q-1} B_j(s+h_j) \right] v^*(s)ds$$

$$+ \; C^T \int_{t^*_{1-h_{j+1}}}^{t^*-h_j} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t^* - s + h_j)^{q-1} B_j(s+h_j) \right] v^*(s)\mathbf{ds} \qquad (3.6).$$

$for \; -1 \le s \le 1 \; and \; u, v^* \; \in U, v^*$ **being optimal control.**

$Subtracting \; equation (3.6) \; from \; equation (3.5)\;, we \; have$

$$0 = C^T \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1 - s + h_j)^{q-1} B_j(s+h_j) \right] u^*(s)\,\mathbf{ds}$$

$$+ \; C^T \int_{t_{1-h_{j+1}}}^{t-h_j} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t - s + h_j)^{q-1} B_j(s+h_j) \right] u^*(s)ds$$

$$- \; C^T \int_{t^*_{1-h_{j+1}}}^{t^*_{1-h_j}} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1{}^* - s + h_j)^{q-1} B_j(s+h_j) \right] v^*(s)ds$$

$$+ \; C^T \int_{t^*_{1-h_{j+1}}}^{t^*-h_j} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t^* - s + h_j)^{q-1} B_j(s+h_j) \right] v^*(s)\mathbf{ds}$$

$$0 = C^T \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1 - s + h_j)^{q-1} B_j(s+h_j) \right] [u^*(s)ds - v^*(s)]ds$$

$$+ \, C^T \int_{t_{1-h_{j+1}}}^{t-h_j} \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t - s + h_j)^{q-1} B_j(s+h_j) \right] [u^*(s)ds - v^*(s)]ds$$

$$\Rightarrow 0 = \left\{ \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} (t_1 - s + h_j)^{q-1} B_j(s+h_j) \right] + \left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} (t - s + h_j)^{q-1} B_j(s+h_j) \right] \right\} \{u^*(s) - v^*(s)\}$$

$\Rightarrow \mathbf{0} = u^*(s) - v^*(s), \Rightarrow u^*(s) = v^*(s).$

This establishes the uniqueness of the optimal control for the system (1.1).

IV. CONCLUSION

We have established necessary and sufficient conditions for the system- Impulsive Quasilinear Fractional Mixed Volterra-Fredholm Type Integro-differential Equations in Banach spaces with Multiple Delays in the Control, to have an optimal control.We have also established the relationship between the Relative Controllability of our system and the Intersection of its two set functions namely- Attainable set and Targes set showing that if. A(t_1,t_0)∩G(t_1,t_0)≠∅ for t∈[t_0 ,t_1] , then the system is relatively controllable and an optimal control for the system exists .

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