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Construction of Rook Polynomials Using Generating Functions and $n \times m$ Arrays

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***Abstract*— In this paper, we studied the game of chess, the rook and its movements to capture pieces in the same row or column as the rook. With this idea we applied it to combinatorial problems which involve permutation with Forbidden positions. By applying generating functions and $n \times m$ arrays to construct rook polynomials for a board that is decomposes into n disjoint sub-boards $B_1, B_2, \ldots, B_n$ and a rook polynomials with $(\mathfrak{M}, R_{0,\infty})$-possible rook movement spaces in a combinatorical way.**

***Index Terms*— r-arrangement, combinatorial structures, Chess movements, Lebesgue measure, µ-integral.**

I. Introduction

A rook polynomial is the generating function to determine the number of ways to put a non-attacking rooks on a generalized board. However, the rook positioning on a board to be used very broadly. As an example, suppose we have this board as the board of possible rook movement.

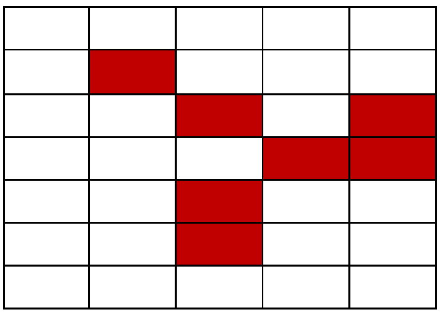

Fig. 1

Where the darkened squares are the rook "Forbidden positions." Then, we can put a rooks on a board in 1 way. In this case, one rook can be put anywhere, and there are exactly two ways to place two rooks on the board. Thus, the general rook polynomial is given by;

$$1 + xr_1(B_1) + x^2r_2(B_1) + \ldots + x^nr_n(B_1).$$

Let r_0 be the number of ways to place a rooks. This number is always 1. We let r_k be the number of ways to place k rooks. The rook polynomial is

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$$r_0 + xr_1(B_1) + x^2r_2(B_1) + \ldots + x^nr_n(B_1) + \ldots.$$

This looks like an infinite series but it only has finitely many terms, making it a polynomial, since there can't be more rooks than rows or columns in the board.

Many authors have studied other related problem with different techniques and proved their results for 2-arrangements of rooks on rows or columns in the board of chess. The several authors (see e.g. Jay, (2000), Heckman, (2006), Feryal, (2013), Gessel, (2013) , Pyzik, (2013) and John, (2013)) addressed the rooks on rows or columns in the board of chess while studying the problem of rook polynomial on a chess board with restrictions, such as the board of darkened squares (fig. 2).

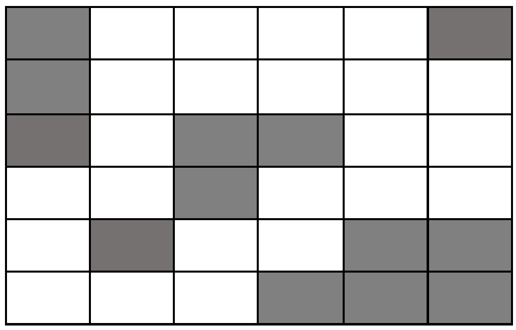


Fig. 2

However, the problem of counting arrangements of objects where there are restrictions in some of the positions in which they can be placed has been addressed by different authors with applications such as; need to match applicants to jobs, where some of the applicants cannot hold certain jobs, or need to pair up artists, but some of the artists cannot be paired up with some of the artists [5]. Now, to address this type of problems where there is need to find the number of arrangements with "Forbidden positions" has been addressed so many years ago. These can be traced back to the early eighteenth century when the French mathematician PIERRE DE MONTMORT studied the problem des rencontres (the matching problem) [7, 11]. In this paper our focus is on forbidden positions in a chess board with n-disjoint sub-boards and a rook polynomials with $(\mathfrak{M}, R_{0,\infty})$-possible rook movement spaces where a rook has the ability to capture pieces in the same row or column as the rook. We use this idea and apply it to combinatorial problem that involves permutation with forbidden positions.

1.1 Basic definitions

Rook: A rook is a chess piece that moves horizontally or vertically and can take (or capture) a piece if that piece rests on a square in the same row or column as the rook [5, 7, 8].

a. Board: A board B is an $n \times m$ array of n rows and m columns. When a board has a darkened square, it is said to have a forbidden position.
b. Rook polynomial: A rook polynomial on a board B, with forbidden positions is denoted as $R(x,B)$, given by $R(x,B) = \sum_{i=1}^{k} r_i(B)x^i$

Where $R(x,B)$ has coefficients $r_i(B)$ representing the number of non-capturing rooks on board B. Clearly, we have just one way of not placing a rook. Thus $r_0(B) = 1$

c. A board B with forbidden positions, is said to be disjoint if the board can be decomposed into two sub-boards $B_i : i = 1 \text{ and } 2$ such that, neither B_1 nor B_2 share the same row or column. [8]

Fig. 3

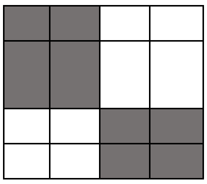
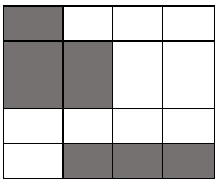


Fig. 4



Clearly, fig 3 is disjoint while fig 4 is not.

Boards are invariant, they can be rearranged by swapping rows with rows or by swapping columns with columns. This allows us to attempt to make non-disjoint boards into disjoint boards. Non-taking rooks is to enumerate the number of ways of placing *i*-rooks on a chessboard such that no rook will be captured by any other rook.

1.2 Derangement (see e.g. Kenneth, 2012)

The number of derangements of a set with n elements is given by;

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

2.1Principle of inclusion-exclusion (PIE). [10, 11]

If $(B_1, B_2, B_3, \ldots, B_k)$ is any sequence of finite sets, then

$$n\left(\bigcup_{i=1}^{k} B_i\right) = \sum_{I \subset |k| / I \neq \emptyset}^{k} (-1)^{n(I)-1} n\left(\bigcap_{i \in I} B_i\right)$$

Where I is an indexing set and $n(I)$ is the cardinality of the indexing set.

2.2Decomposes into two disjoint sub-boards [9]

If B is a board of darkened squares that decomposes into two disjoint sub-boards $B_i : i = 1 \text{ and } 2$, then $R(x,B) = R(x,B_1)R(x,B_2)$. [10]

2.3 n-Objects among m positions [7, 8]

The number of ways to arrange n objects among m positions $(m \geq n)$ when there are restricted positions is

$$R(x,B)P_{(m,n)} = P_{(m,n)} - r_1(B)P_{(m-1,n-1)} + r_2(B)P_{(m-2,n-2)} - \ldots + (-1)^m r_m(B)P_{(m-n,0)}$$

When $m = n$

$$\sum_{i=0}^{n} (-1)^i r_i(B) P_{(m-i,n-i)} = P_{(n,n)} - r_1(B)P_{(n-1,n-1)} + r_2(B)P_{(n-2,n-2)} + \ldots (-1)^n r_n(B)P_{(n-i,n-i)}$$

Theorem 2.4

Let B be a board of darkened squares. Let S be one of the squares of B, and let B_s and B_s^* [6]. Then

$$R(x,B) = R(x,B_s) + R(B_s^*)x$$

2.5 A simple matching polynomials [7]

Let us take S_n to be the set of complete matchings' of [n]: partitions of [n] into blocks of size 2. Then $S_n = 0$ if n is odd and if $n = 2i$ then

$$M_n = (n-1)! = (n-1)(n-3)\ldots 1 = \frac{(2i)!}{2^i i!}.$$

The properties that we consider are of the form "$\{i,j\}$ is a block." Here if A is a set of compatible properties then $\rho(A) = 2|A|$, and the linear functional function Φ has the integral representation

$$\Phi(f(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} f(x)dx,$$

The matching polynomials for "complete boards" are the Hermite polynomials [7]

$$H_n(x) = \sum_{i=0}^{n} \frac{(-1)^i n! x^{n-i}}{2^i i! (n-2i)!} = R(x,B)$$

and these are easily seen to be orthogonal combinatorially

II. Main Results

Theorem 3.0 (Restricted Positions)

The number of ways to arrange n objects among m positions $(m \geq n)$ such order is maintain, when there are restricted positions is

$$R(x,B) = \sum_{i=0}^{n} \frac{(-1)^i r_i(B) \binom{m-i}{n}}{\binom{m}{n}}$$

When $m = n$

$$R(x,B) = \sum_{i=0}^{n} (-1)^i r_i(B) \binom{n-i}{n}$$

Proof

Theorem 3.0 follows from the proof of theorem 2.3, following the method from the first argument of the m elements we have

$$1 - \frac{r_1(B)\binom{m-1}{n}}{\binom{m}{n}} + \frac{r_2(B)\binom{m-2}{n}}{\binom{m}{n}} - \cdots (-1)^i \frac{r_i(B)\binom{m-i}{n}}{\binom{m}{n}}$$

$$= \sum_{i=0}^{n} \frac{(-1)^i r_i(B)\binom{m-i}{n}}{\binom{m}{n}} = R(x,B)$$

Theorem 3.2 (n-disjoint sub-boards)

If B is a board of darkened squares that decomposes into n disjoint sub-boards $B_1, B_2, \ldots B_n,$ then $R(x,B) = R(x,B_1)R(x,B_2) \ldots R(x,B_n)$.

Proof,

Let

$$R(x,B_1) = \sum_{i=0}^{n} (x)^i r_i(B_1) = 1 + x r_1(B_1) + x^2 r_2(B_1) + \ldots + x^n r_n(B_1)$$

$$R(x,B_2) = \sum_{i=0}^{n} (x)^i r_i(B_2) = 1 + x r_1(B_2) + x^2 r_2(B_2) + \ldots + x^n r_n(B_2)$$

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$$R(x,B_n) = \sum_{i=0}^{n} (x)^i r_i(B_n) = 1 + x r_1(B_n) + x^2 r_2(B_n) + \ldots + x^n r_n(B_n)$$

Then

$$R(x,B) = R(x,B_1)R(x,B_2) \ldots R(x,B_n)$$

$$= \sum_{i=0}^{n} \prod_{k=0}^{n} \mathcal{X}_{B_j,K}(x)^i r_i(B_j), \quad j = 1,2,\cdots,n$$

The nth coefficient of $R(x,B) = R(x,B_1)R(x,B_2) \ldots R(x,B_n)$ is

$$r_0(B_1)r_n(B_2)r_{n-1}(B_3) \ldots r_{n-i+1}(B_n) + r_1(B_1)r_{n-2}(B_2)r_{n-3}(B_3) \ldots r_{n-i}(B_n) + \cdots + r_n(B_1)r_{n-1}(B_2)r_{n-2}(B_3) \ldots r_0(B_n)$$

This shows that when there is no rook on B_1 there are n rooks on B_2 and n-1 rooks on B_n. This continues until we have n rook on B_1, there are no rooks on B_2 and 1 rooks on B_n. Thus

$$R(x,B_1)R(x,B_2) \ldots R(x,B_n) = \sum_{i=0}^{n} \prod_{k=0}^{n} \mathcal{X}_{B_j,K}(x)^i r_i(B_j) = R(x,B)$$

Theorem 3.3 (Possible rook movements)

Let B be a board of darkened squares. Let $R: B \to [0,\infty]$ be $(\mathfrak{M}, R_{0,\infty})$-possible rook movements. Then, there exist simple rook movement functions $\gamma_n, n \in \mathbb{N}_+ on\ B$ such that $\gamma_n \uparrow R$.

Proof

Given the rook movement $n \in \mathbb{N}_+$, set $E_{in} = R^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right[\right), for\ all\ i \in \mathbb{N}_+$ and

$$\lambda_n = \sum_{i=1}^{\infty} \frac{i-1}{2^n} \mathcal{X}_{E_m} + \infty \mathcal{X}_{R^{-1}}(\{\infty\})$$.

It is clear that $\lambda_n \leq R$ and that $\lambda_n \leq \lambda_{n+1}$. Then, set $\gamma_n = \min(n, \lambda_n)$ and the result follows.

Let $(B, \mathfrak{M}, \mu)$ be a positive rook movement space and $\gamma: B \to [0,\infty[$ a simple rook movement function. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the distinct values of the simple function γ and if

$$E_i = \gamma^{-1}(\{\alpha_i\}), for\ all\ i = 1,2,\ldots,n,$$

then we have

$$\gamma = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{E_i}$$

Furthermore, if $A \in \mathfrak{M}$ we define

$$v(A) = \int_A \gamma d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i \cap A) = \sum_{i=1}^{n} \alpha_i \mu^{E_i}(A)$$

It follows that this formula still holds if $(E_i)_i^n$ is a possible rook movement partition on B and $\alpha_i = \alpha_i$ on E_i for each $i = 1,2,\ldots,n$. Clearly v is a positive rook movement since each move in the right side is a positive rook movement as a function of A. Thus

$$\int_A \alpha\gamma d\mu = \alpha \int_A \gamma d\mu, \qquad if\ 0 \leq \alpha < \infty$$, and

$$\int_A \gamma d\mu = \alpha\mu(A), \qquad if\ \alpha \in [0,\infty[$$, and γ is a simple rook movement function such that $\gamma = \alpha$, on A. If ϑ is another simple rook movement function and $\gamma \leq \vartheta$

$$\int_A \gamma d\mu = \int_A \vartheta d\mu.$$

To see this, let $\beta_1, \beta_2, \ldots, \beta_t$ be the distinct values of ϑ and $D_j = \vartheta^{-1}(\{\beta_j\}), \quad j = 1,2,\ldots,t$

Then, putting $\mathfrak{B}_{ij} = E_i \cap D_j$,

$$\int_A \gamma d\mu = v\left(\cup_{ij}\left(A \cap \mathfrak{B}_{ij}\right)\right) = \sum_{ij} v\left(A \cap \mathfrak{B}_{ij}\right) = \sum_{ij} \int_{A \cap \mathfrak{B}_{ij}} \gamma d\mu$$

$$= \sum_{ij} \int_{A \cap \mathfrak{B}_{ij}} \alpha_i d\mu \le \sum_{ij} \int_{A \cap \mathfrak{B}_{ij}} \beta_j d\mu = \int_A \vartheta d\mu$$

Similarly, we can show that

$$\int_A (\gamma + \vartheta) d\mu = \int_A \gamma d\mu + \int_A \vartheta d\mu$$

From the above it follows that

$$\int_A \gamma \mathcal{X}_A d\mu = \int_A \sum_{i=1}^{n} \alpha_i \mathcal{X}_{E_i \cap A} d\mu = \sum_{i=1}^{n} \alpha_i \int_A \mathcal{X}_{E_i \cap A} d\mu = \sum_{i=1} \alpha_i \mu(E_i \cap A)$$

And $\int_A \gamma \mathcal{X}_A d\mu = \int_A \gamma d\mu.$

If $f: B \to [0, \infty]$ is an $(\mathfrak{M}, R_{0,\infty})$ -possible rook movements function and $A \in M$, we can define;

$$\int_A f d\mu = sup\left\{\int_A \gamma d\mu, \quad 0 \le \gamma \le f,\ \gamma \text{ is simple rook movement}\right\} \quad (i)$$

$$\int_A f d\mu = sup\left\{\int_A \gamma d\mu, \quad 0 \le \gamma \le f,\ \gamma \text{ is simple rook movement and } \gamma = 0 \text{ on } A\right\} \quad (ii)$$

However, the left hand side of the equation is called the Lebesgue integral of f over A or the μ -integral of f over A. Thus, definitions (i) and (ii) of the Lebesgue integral of γ over A of a simple rook movement function $\gamma: B \to [0, \infty[$ over A also correspond.

III. NUMERICAL APPLICATIONS

Example 4.1

Let B be a board of darkened squares. Let $R: B \to [0, \infty]$ be the possible rook movements and $\int_B R d\mu < \infty$. Then, there exist simple rook movement functions $\{R = \infty\} = R^{-1}(\{\infty\}) \in Z_\mu$.

Solution

Applying the Markov Inequality, we have

$$\mu(R = \infty) \le \mu(R \ge \alpha) \le \frac{1}{\alpha} \int_B R d\mu$$

For each $\alpha \in]0, \infty[$. Thus, $\mu(R = \infty) = 0.$ The results follows.

Example 4.2

The construction of Chukwuemeka Odumegwu Ojukwu University design it's faculties into plots for the following areas A, C, F, G and K. To offer this plots to any of the following faculties; Biological Science (B), Engineering (E), Law (L), Management (M) and Physical Science. It follows that; a. the Faculty of Biological Science (B) can't beallocated to plotF and G

b. the Faculty of Engineering (E) can't be allocated to plot A and C,

c. the Faculty of Law (L) can't be allocated to plot F

d. the Faculty of Management (M) can't be allocated to plot F,

e. the Faculty of Physical Science can't be allocated to plot C and K

In how many ways can the University authorities construct twenty Faculties with this same plot arrangement?

Solution

The construction site for the University

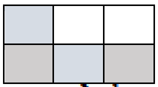
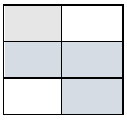


Fig. 7



The selected areas are; A, C, F, G and K from fig 7 and the faculties are; Biological Science (B), Engineering (E), Law (L), Management (M) and Physical Science. Wherein m = 5 and n = 5. We can construct a board of order 5×5 with darkened squares for the Forbidden positions.

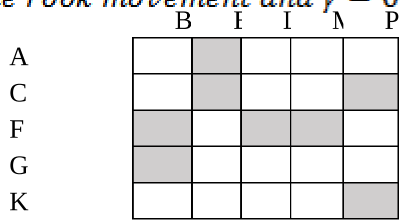
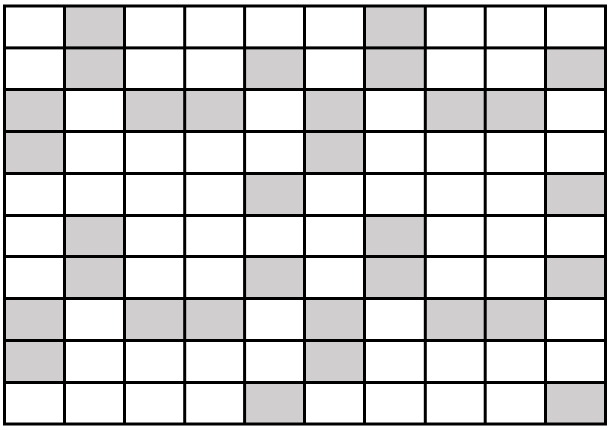


Fig. 5

We now construct a new board with the following row



movements F-k and K-F, and columns movements E-M and M-E respectively. Thus, arranging the board into disjoint sub-boards B_1 and B_2 we have;

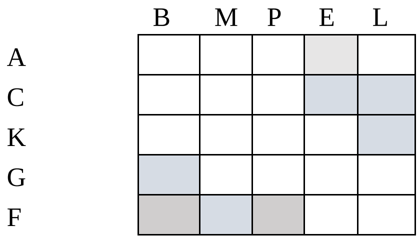


Fig. 6

We have the following row movements FK and KF, the columns movements EM and ME. That generate the two sub-boards $B_1$and B_2 respectively.

Applying the definition 1.1 we have that $r_0(B_1) = r_0(B_2) = 1$ and it also follows that $r_1(B_1) = r_1(B_2) = 4$ and $r_2(B_1) = 2, \; r_2(B_2) = 3$

Therefore the Rook polynomial on board B, with eight forbidden positions can now be simplified as follows.

$$R(x,B_1) = \sum_{i=1}^{2} r_i(B_1)x^i = 1 + 4x + 2x^2$$

$$R(x,B_2) = \sum_{i=1}^{2} r_i(B_2)x^i = 1 + 4x + 3x^2$$

$$R(x,B) = \sum_{i=1}^{2} r_i(B_j)x^i = (1+4x+2x^2)(1+4x+3x^2), for\ all\ j = 1,2$$

$$= 1 + 8x + 21x^2 + 20x^3 + 6x^4$$

However, the University authorities require twenty Faculties with the same construction design. We shall have $4(5 \times 5)$ order plots and applying theorem 3.2 for 4 disjoint sub-plots, we have

$$R(x,B) = \sum_{i=0}^{2} \prod_{k=0}^{4} x_{B_j,K}(x)^i r_i(B_j)$$

$$= (1+8x+21x^2+20x^3+6x^4)...(1+8x+21x^2+20x^3+6x^4)$$

$$= 1 + 16x + 106x^2 + 376x^3 + 773x^4 + 936x^5 + 653x^6 + 240x^7 + 36x^8$$

$$R(x,B)P_{(5,5)} = \sum_{i=0}^{2} \prod_{k=0}^{4} x_{B_j,N} r_i(B_j) P_{(5,5)}$$

$$R(x,B)P_{(5,5)} = P_{(5,5)} + 8P_{(4,4)} + 21P_{(3,3)} + 20P_{(2,2)} + 6 = 484\ ways$$

The University authorities has 1936 ways of constructing twenty Faculties with this same plot arrangement.

Thus, fig.8the Universityconstruction design for the twenty Faculties.

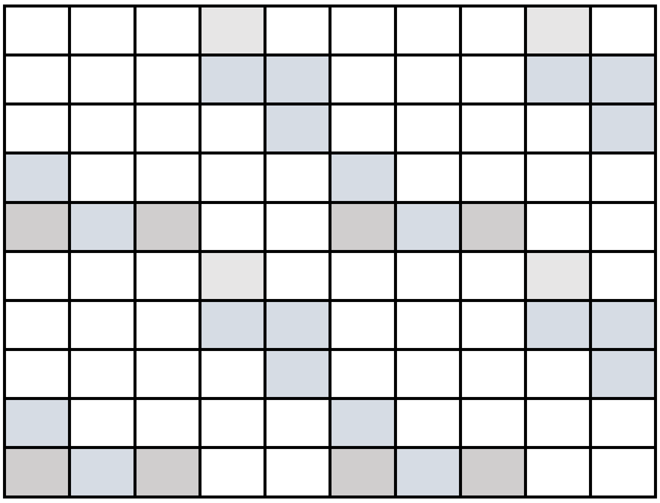


Fig. 7

IV. Conclusion

The results are generalizations of the generating functions and $n \times m$ arrays to construct rook polynomials $R(x,B)$ for ordered restricted positions, a board of darkened squares that decomposes into n disjoint sub-boards $B_1, B_2, \ldots B_n$ and $(\mathfrak{M}, R_{0,\infty})$-possible rook movementson a chess board of darkened squares.

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