On the Eigenvalue Problem of the Dirac Hamiltonian in a Singular Spacetime Background

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***Abstract*— We studied the motion of a spin onehalf particle in a singular spacetime metric of exponential type. This singular spacetime metric is obtained from a particular interpretation of the Newtonian gravitational force. For a given energy, the motion of a particle is described by the eigenvalue modes of the Dirac operator in a singular space time metric of exponential type. In this paper only the first order solution is considered leading to a cubic algebraic equation for energy eigenvalue. A novel interpretation for a gravitational atom is obtained.**

***Index Terms*— Eigenvalue Problem, Newtonian gravitational force.**

I. Introduction

The motion of a classical particle in a static, spherically symmetric spacetime of the exponential type is described [1] by the classical Hamiltonian

$$H = e^{+u}\sqrt{p_i p^i c^2 + m_0^2 c^4}$$

(1) where $p^i = e^{-2u} p_i = m_0 v^i, i = 1,2,3$. The compatability of the background spacetime metric with the mass-energy relation of Einstein,

$$d(mc^2) = v^i dp_i = v^i F_i dt = mc^2 du \quad (2)$$

determines both the form of the function $u = R/r$, and the form of the gravitational force, $F_i = -\partial_i H$, where R is the gravitational radius of the source of spacetime metric.

The motion of a Dirac particle in this static, spherically symmetric spacetime metric of the exponential type is described by the corresponding wave equation

$$(-ic\hbar\alpha^k e^{-u} D_k + \beta m_0 c^2)\psi = e^u E \quad (3)$$

obtained by linearising the Hamiltonian (1) using Dirac's (α^k, β)-matrices. Here D_k denotes the covariant differential operator acting on the vector wave function ψ, according to

$$D_k\psi^\mu = \partial_k\psi^\mu + \Gamma^\mu_{k\nu}\psi^\nu, \mu,\nu = 0,1,2,3. \quad (4)$$

where $\Gamma^\mu_{k\nu}$ is the Levy-Civita connection. After introducing the metric depended γ -matrices, $\gamma^0 = e^u\beta, \gamma^k = e^{-u}\beta\alpha^k$, satisfying the anti-commutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu} \quad (5)$$

with the following components of the metric tensor $g^{\mu\nu} : g^{00} = e^{2u}, g^{kk} = e^{-2u}, g^{0k} = 0$, we can rewrite the Dirac equation (3) in the covariant form [2]

$$(-i\hbar\gamma^\mu D_\mu + m_0 c)\psi = 0 \quad (6)$$

where $-i\hbar D_0\psi = -i\hbar\partial_t\psi = E\psi$.

In the next Section, we shall try to solve (3) for discrete energy levels.

II. Eigenvalue problem

It is not difficult to see that the equation (3) can be put in the standard form of two coupled first order differential equations ($\hbar = c = 1$)

$$F'(r) + \Gamma^1_{11}F(r) + e^{u/2}\frac{\kappa}{r}F(r) = (e^u m_0 - e^{2u}E)G(r),$$

$$G'(r) + \Gamma^2_{12}G(r) - e^{u/2}\frac{\kappa - 1}{r}G(r) = (e^u m_0 + e^{2u}E)F(r)$$

(8)

where $\kappa = j + 1/2$. In deriving the equations (7,8), we have used the standard decomposition of the wave function $\psi^T = (FY^+_{jlm_j}, iGY^-_{jlm_j})/r$ with the following properties

$$\beta(\Sigma^k L_k + \hbar)\psi = -\kappa\psi,$$

$$\sigma_r Y^\pm_{jlm_j} = -Y^\mp_{jlm_j}$$

where $i\vec{\Sigma} = \vec{\alpha}\times\vec{\alpha}$ and $\sigma_r = \vec{\sigma}\cdot\vec{r}/r$ are operators and acting only on the angular momentum parts of the wave function. We note that isotropic spacetime metric has modified mass and energy of the Dirac particles, making the eigenvalue problem particularly difficult to solve.

The eigenvalue problem is then reduced to solving the two algebraic equation for s and E as a function of the radial quantum number n under condition that $s > 0$.

In a curved spacetime background is given by [2]

$$(-i\gamma^\mu(x)D_\mu + mc)\psi(x) = 0 \quad (11)$$

In the equations (7) i (8), we expand exponential functions into the power series of R/r keeping the same powers of

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$1/r$ on both sides of the equations. By neglecting all terms of order higher then $1/r$ we obtain:

$$\alpha\left(\dot{F}(\rho)+\frac{\kappa}{\rho}F(\rho)\right) = (m_0 - E + \alpha\frac{R}{\rho}(m_0-2E))G(\rho) \quad (12)$$

$$\alpha\left(\dot{G}(\rho)-\frac{\kappa-1}{\rho}G(\rho)\right) = (m_0 + E + \alpha\frac{R}{\rho}(m_0+2E))F(\rho) \quad (13)$$

We assume that the regular solution at $r=0$ and at $r=\infty$ of (12,13), is of the form $F(\rho)=\rho^s e^{-\rho}A_n(\rho), G(\rho)=\rho^s e^{-\rho}B_n(\rho)$, where $\rho=\alpha r$ and A_n, B_n are two n-th order polynomials:

$$A_{n'} = \sum_{n\le n'} a_n \rho^n \quad \text{and} \quad B_{n'} = \sum_{n\le n'} b_n \rho^n$$

where $a_{n'} \neq 0, b_{n'} \neq 0$

By comparing the coefficient for $n=n'$ we get:

$$\alpha^2 = m_0^2 - E^2$$

By equating the coefficients of ρ^{-1} and $\rho^{n'-1}$ we obtain the following system of two equations:

$$R^2(m_0^2-4E^2)-\left(s+\frac{1}{2}\right)^2+J^2=0$$

$$R(m_0^2-2E^2)+\alpha\left(s+n'+\frac{1}{2}\right)=0$$

By eliminating s from (15) and (16) and using (14) we obtain a two cubic equations in the variables α and β

$$\alpha^2 = m_0^2 - E^2,\ \beta = s+\frac{1}{2},\ \beta>\frac{1}{2}$$

$$4Rn'\alpha^3 + (m_0^2R^2 + J^2 - n'^2)\alpha^2 - 2n'm_0^2R\alpha - m_0^4R^2 = 0 \quad (17)$$

$$2n'\beta^3 + (m_0^2R^2 + J^2 + n'^2)\beta^2 + 2n'(3m_0^2R^2 - J^2)\beta + (3m_0^2R^2 - J^2)n'^2 - (m_0^2R^2 - J^2)^2 = 0$$

According to a Theorem [3] : The polynomial p has exactly m($\le$n) distinct zeroes, all of which are real, if and only if

$\delta_1(p)>0, \delta_3(p)>0,..., \delta_{2m-1}(p)>0$

and $\delta_j(p)=0,\ j>2m$

where for $p=a_0z^n+a_1z^{n-1}+...+a_n$, $\delta_1(p)$ is the j-th principal minor of the matrix:

$$D_{2n}(p)=\begin{pmatrix} na_0 & (n-1)a_1 & (n-1)a_2 & \dots & a_{n-1} & 0 & \dots & 0 & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & \dots & 0 & 0 \\ 0 & na_0 & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & \dots & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & \dots & 0 & 0 \\ 0 & 0 & na_0 & \dots & 3a_{n-3} & 2a_{n-2} & \dots & 0 & 0 \\ 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & na_0 & (n-1)a_1 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}$$

n=3 and m=1

$$f(z) = a_0z^3 + a_1z^2 + a_2z + a_3$$

$$D_6(p)=\begin{pmatrix} 3a_0 & 2a_1 & a_2 & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \end{pmatrix}$$

$$\delta_1(p)=3a_0>0, \delta_2(p)=a_1a_0,..., \delta_3(p)=2a_0(a_1^2-3a_0a_2), \delta_5(p)=discrim(p)$$

Discriminant of the equation (17) is equal to

$$\Delta=4m_0^2R^2[(J^6+(3m_0^2R^2-2n^{'2})J^4+(3R^4m_0^4+32R^2n^{'2}m_0^2+n^{'2})J^2+R^2m_0^2(R^4m_0^4-3n^{'4}-74R^2n^{'2}m_0^2)]$$

In case $n^{'}=0$, $J=\frac{1}{2}$ we have

$\Delta=\frac{1}{64}(4R^2m_0^2+1)^3>0$, $m_0^2R^2>\frac{3}{4}\Rightarrow\exists$ one real solution

For $n^{'}=1$, $J=\frac{1}{2}$, $m_0^2R^2>6{,}75$,

$$\Delta=\frac{1}{16}R^2m_0^4(64R^6m_0^6-4688R^4m_0^4+332R^2m_0^2+9)>0$$

In interval $(6.75,73,179)$ $\exists$ 3 real solutions. Out of interval exists only one solution.

For $n^{'}=2$, $J=\frac{1}{2}$, $m_0^2R^2>18{,}88$

$$\Delta=\frac{1}{16}R^2m_0^4(64(Rm_0)^6-18896(Rm_0)^4-1012(Rm_0)^2+225)>0$$

In interval $(18.88,2953)$ $\exists$ 3 real solutions. Out of interval exists only one solution.

III. CONCLUSION

In this paper we have studied the eigenvalue problem of the Dirac Hamiltonian in a singular spacetime background. From the mathematical point of view the problem is presently unsolvable except in an approximate way when $R<<r$. In this case the eigenvalue problem is reduced to solving the bicubic algebraic equations in the energy variable. We have found that bound states depend crucially on the Rm_0 product. We have also found the region where exists three real solutions for the first bound state level for which we do not have a clear physical interpretation.

REFERENCES

[1] M.Martinis,On the gravitational energy shift for matter waves,arXiv:1004.0826, Is exponential metric a natural space-time metric of Newtonian gravity?,arXiv:1009.6017

[2] F. Rosenblat, Phys.Rev.Lett.,**44** (1980)1559.

[3] Olga Holtz, Mikhail Tyaglov, Structured matrices, continued fractions, and root localization of polynomials, SIAM Review,54(2012), no.3, 421-509 or arXiv:0912.4703v3 (math.CA)

[4] Huseyin Yilmaz , Einstein, the exponential metric, and a proposed gravitational Michelson-Morley experiment - Hadronic J. 2 (1979) 997-1020

[5] Huseyin Yilmaz ,Gravity and Quantum field theory,Annals of the New York Academy of Sciences, Volume 755, Fundamental Problems in Quantum Theory pages 476 -499, April 1995

[6] Charles W. Misner,Yilmaz Cancels Newton, arXiv:gr-qc/9504050

[7] Carroll O. Alley,The Yilmaz Theory of Gravity and Its Compatibility with Quantum Theory, Annals of the New York Academy of Sciences, Volume 755, Fundamental Problems in Quantum Theory pages 464 -475, April 1995