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A Second Order Fuzzy Differential Equation for The Case of A Semi-Confined Aquifer

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***Abstract*-Differential equations are encountered very often in engineering problems and generally in al sciences. Modeling the effect of variation of physical quantities such as temperature, pressure, velocity, stress, strain, current moisture and many other on engineering problems requires most of the times the establishment of differential equations. For simplicity reason the parameters and variables involved which are measured or estimated from experience are considered exact even though they often contain uncertainties. One way do deal with these uncertainties nowadays is through convex fuzzy sets. According to all the above it is almost unavoidable to introduce fuzzy parameters and variables in the solution of differential equations. Much research was carried out during the recent years in theoretic and applied subjects containing fuzzy differential equations with H-derivative. This method though, in some cases has some disadvantages leading to solutions with increasing support as time t increases. In order to alleviate this disadvantage the generalized differentiability (G-H derivative) was introduced. In this paper the case of a semi-confined aquifer is studied, which is bounded on top by a thin semi-permeable layer and on bottom by an impermeable layer. This system leads to a second order differential equation with fuzzy boundary. The solution of this problem is obtained using the generalized H-derivative.**

***Index Terms*— Fuzzy applications, fuzzy differential equations, semi-confined aquifer.**

I. Introduction

Differential equations are encountered very often in engineering problems and generally in al sciences. Modeling the effect of variation of physical quantities such as temperature, pressure, velocity, stress, strain, current moisture and many other on engineering problems requires most of the times the establishment of differential equations. Also, the impact of certain physical quantities on other physical quantities leads to differential equations. So, the theory of differential equations is found in mechanical vibration or structural dynamics, heat transfer, hydraulics etc. The engineers should be able to model actual problems using mathematical equations and through their solution to comprehend the behavior of the systems under consideration. Differential equations are extensively used in mathematical modeling and engineering applications. Theory and techniques for solving differential equations are applied to solve practical engineering problems.

For the sake of simplicity the parameters and variables involved in the systems are considered as crisp or defined exactly. Real problems though lack exact information about the variables and parameters obtained by experiment or experience. Nowadays, these uncertainties are modeled through convex normalized fuzzy sets. So, it is necessary to include in the solution of differential equations fuzzy variables and fuzzy parameters.

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing subjective information in mathematical models. The development of this subject has been focused to theoretical topics ([1]-[5]-[14]-[16]) as well as to applications like population models, civil engineering and hydraulics ([11]-[12]).

Initially differentiable fuzzy functions were studied by [18], who generalized and extended the concept of [13], differentiability (H-derivative) of set valued mappings to the class of fuzzy mappings. Also, [14] and [19] developed a theory for fuzzy differential equations.

Many works have been carried out during last years in theoretical and applied topics for fuzzy differential equations with H-derivative ([14]-[16]-[19]-[21]). But in some cases this method suffers certain disadvantages that lead to solutions with increasing support as time t increases ([5]-[9]). This proves that in some cases this solution is not a good generalization of the corresponding crisp case. In order to surpass the above deficiency, the generalized differentiability was introduced ([2]-[3]-[4]). This new derivative is defined for a larger class of fuzzy functions than Hukuhara derivative [13].

In this paper the case of a two-point fuzzy boundary value problem for a second order fuzzy differential equation is studied, the problem concerns the case of a semi-confined aquifer which is bounded on top by a thin semi-permeable layer and on bottom by an impermeable layer. This system leads to a second order differential equation with fuzzy boundaries. The solution of this problem is obtained using the generalized H-derivative.

II. MATHEMATICAL MODEL

A. Definitions

Definition 1. A fuzzy set $\tilde{U}$ on a universe set X is a mapping $\tilde{U}: X \rightarrow [0,1]$, assigning to each element $x \in X$ a degree of membership $0 \leq \tilde{U}(x) \leq 1$. The membership function is also defined as $\mu_{\tilde{U}}(x)$ with the properties:

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$\mu_{\tilde{U}}$ is upper semi continuous,

$\mu_{\tilde{U}}(x)$ =0, outside of some interval [c, d],

there are real numbers a and b , $c \le a \le b \le d$, such that $\mu_{\tilde{U}}$ is increasing on [c,a], decreasing on [b, d] and $\mu_{\tilde{U}}(x)$ =1 for each $x \in [a,b]$.

$\tilde{U}$ is a convex fuzzy set (i.e. $\mu_{\tilde{U}}(\lambda x + (1-\lambda)x) \ge \min\{\mu_{\tilde{U}}(\lambda x), \mu_{\tilde{U}}((1-\lambda)x)\}$

Definition 2. Let X being a Banach space and $\tilde{U}$ being a fuzzy set on X. We define the α-cuts of $\tilde{U}$ as $[\tilde{U}]^\alpha = \{x \in X \big| \tilde{U}(x) > 0\}$, with $0 \le \alpha \le 1$, and for α=0, we also define the closure as $[\tilde{U}]^0 = \{x \in X \big| \tilde{U}(x) > 0\}$.

Definition 3. Let Κ(X) the family of all nonempty compact convex subsets of a Banach space. A fuzzy set $\tilde{U}$ on X is called compact if $[\tilde{U}]^\alpha \in K(X),\ \forall \alpha \in [0,1]$. The space of all compact and convex fuzzy sets on X is denoted as F (X).

Definition 4. Let $\tilde{U} \in$ F (R). The α-cuts of $\tilde{U}$, are: $[\tilde{U}]^\alpha = [U_\alpha^-(x),\ U_\alpha^+(x)]$.

According to the representation theorem of [15] and the theorem of [10] the membership function and the α-cut form of a fuzzy number $\tilde{U}$ are equivalent and in particular the α-cuts $[\tilde{U}]^\alpha = [U_\alpha^-(x),\ U_\alpha^+(x)]$ uniquely represent $\tilde{U}$, provided that the two functions are monotonic (U_α^- increasing, U_α^+ decreasing) and $U_1^- \le U_1^+$, for α=1. The derivatives of each α-cut with respect to x, for a given $\alpha \in [0,1]$, are denoted as:

$$(U_\alpha^-)'_x = \frac{\partial U_\alpha^-(x)}{\partial x},\ (U_\alpha^+)'_x = \frac{\partial U_\alpha^+(x)}{\partial x}.$$

Definition 5. The following notations are used for the above derivatives with respect to α

$$\delta((U_\alpha^-)'_x) = \frac{\partial}{\partial \alpha}(\frac{\partial U_\alpha^-(x)}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial U_\alpha^-(x)}{\partial a}) = = (\delta U_\alpha^-)'_x(x),$$

$$\delta((U_\alpha^+)'_x) = \frac{\partial}{\partial \alpha}(\frac{\partial U_\alpha^+(x)}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial U_\alpha^+(x)}{\partial \alpha}) = = (\delta U_\alpha^+)'_x(x)$$

Lemma 1. If both $U_\alpha^-(x)$, and $U_\alpha^+(x)$ are differentiable w.r.t. x, then the α-cuts of the gH (generalized Hukuhara)-derivative of U are:

$$\tilde{U}'_{gH}(x) = [\min\{(U_\alpha^-)'(x),\ (U_\alpha^+)'(x)\},\ \max\{(U_\alpha^-)'(x),\ (U_\alpha^+)'(x)\}],$$

provided that the two functions:

$$\tilde{U}'_{gH}(x)_\alpha^- = \min\{(U_\alpha^-)'(x),\ (U_\alpha^+)'(x)\},$$ and

$$\tilde{U}'_{gH}(x)_\alpha^+ = \max\{(U_\alpha^-)'(x),\ (U_\alpha^+)'(x)\},$$

define a fuzzy number (w.r.t. α). Here the functions $(U_\alpha^-)'$, $(U_\alpha^+)'$ denote the derivatives with respect to x, for given $\alpha \in [0,1]$.

Definition 6. Let $[\tilde{U}] \in$ F(R), with the a-cuts $[\tilde{U}]^\alpha = [U_\alpha^-(x),\ U_\alpha^+(x)]$ satisfying Lemma 1. According to [3]:

1. $\tilde{U}$ is (i)-generalized Hukuhara (gH) differentiable if:

$$(U_1^-)'_x(x) \le (U_1^+)'_x(x)$$

$$(\delta U_\alpha^-)'_x(x) = \frac{\partial}{\partial \alpha}(\frac{\partial}{\partial x} U_\alpha^-(x)) \ge 0$$

$$(\delta U_\alpha^+)'_x(x) = \frac{\partial}{\partial \alpha}(\frac{\partial}{\partial x} U_\alpha^+(x)) \le 0$$

for $\forall \alpha \in [0,1]$.

2. $\tilde{U}$ is (ii)-generalized Hukuhara (gH) differentiable if:

$$(U_1^-)'_x(x) \ge (U_1^+)'_x(x)$$

$$(\delta U_\alpha^-)'_x(x) = \frac{\partial}{\partial \alpha}(\frac{\partial}{\partial x} U_\alpha^-(x)) \le 0$$

$$(\delta U_\alpha^+)'_x(x) = \frac{\partial}{\partial \alpha}(\frac{\partial}{\partial x} U_\alpha^+(x)) \ge 0$$

for $\forall \alpha \in [0,1]$.

B. Seni-confined aquifer

The semi-confined aquifer (Fig.1) is discharging towards a Lake. This aquifer is bounded on top by a thin semi-permeable layer of thickness B′ and on the bottom by an impermeable layer. On top of the semi-permeable layer there is water entrapped by an earth dam. The equation describing the water level variation h as a function of length x is given as:

$$\frac{d^2h}{dx^2} + \frac{h - h_o}{\lambda^2} = 0, \quad (1)$$

where λ is a leakage factor given by the relation:

$$\lambda = \sqrt{KBc}, \quad (2)$$

and c is the hydraulic resistance of the semi-permeable aquifer having time dimensions and is given by the following relation:

$$c = \frac{B'}{K'}. \quad (3)$$

Equation (1) resulted from the application of Darcy's law and also taking into account the leakage from the semi-impermeable layer ([19]-[21]). In relations (1), (2), (3), K is the hydraulic conductivity of the semi-confined aquifer, B is its thickness, B′ is the thickness of the thin semi-permeable layer , K′ is its hydraulic conductivity, h0 is the head of the aquifer for $x \to \infty$, and h1 is the head of the aquifer at x=0.

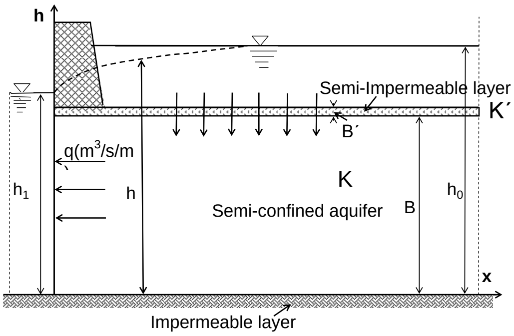

Fig. 1. Semi-confined aquifer.

We introduce now in (1) the drawdown s=ho-h and we obtain:

$$\frac{d^2 s}{dx^2} - \frac{s}{\lambda^2} = 0 \quad . \tag{4}$$

The boundary conditions of the above problem then become:

$$\begin{aligned} s(0) &= s_1 = h_o - h_1, & x &= 0 \\ s(\infty) &= s_o = h_o - h_o = 0 & x &\to \infty. \end{aligned} \tag{5}$$

We replace now in (4):

$$X = \frac{x}{\lambda}, \tag{6}$$

and this leads to :

$$\frac{d^2 s}{dX^2} - s = 0 \quad , \tag{7}$$

with boundary conditions :

$$\begin{aligned} s(0) &= s_0 = h_0 - h_1, & X &= 0 \\ s(\infty) &= s_\infty = h_0 - h_0 = 0 & X &\to \infty. \end{aligned} \tag{8}$$

C. General solution

The above equation has the general solution:

$$s(X, \alpha) = c_1 e^{x/\lambda} + c_2 e^{-x/\lambda} \; .$$

For X=0, $s_0=c_1+c_2$, and for $X \to \infty$, $c_1 = 0$.

So $c_2 = s_0$ and the solution now takes the form:

$$s(X, c) = s_0 e^{-X} \text{ or,}$$
$$h = h_0 - s_0 e^{-X} \; . \tag{9}$$

D. Fuzzy model

We fuzzify now the solution:

$$\tilde{h} = \tilde{h}_0 - \tilde{s}_0 e^{-X}, \quad (10)$$

where $\tilde{s}_0 = (r, m_1)$, $\tilde{s}_\infty = (0, m_2)$, and m1, m2 are the centers of fuzzy numbers and r, 0 are the spreads. The number $\tilde{s}_0$ is a symmetrical triangular fuzzy, while $\tilde{s}_\infty, (m_2 = 0)$, is a crisp number, expressed in fuzzy notation. The fuzzy boundary conditions are now:

$$\begin{aligned} \tilde{s}_0\big|^\alpha &= [s^\alpha{}_{0\ell}, s^\alpha{}_{0r}] = [m_1 - r(1-\alpha),\ m_1 + r(1-\alpha)], \\ \tilde{s}_\infty\big|^\alpha &= [s^\alpha{}_{\infty\ell}, s^\alpha{}_{\infty r}] = [0, 0], (m_2 = 0) \end{aligned} \tag{11}$$

From (9) and (10) we arrive to:

$$\begin{aligned} \tilde{h}\big|^\alpha &= \tilde{h}_0\big|^\alpha - \tilde{s}_0\big|^\alpha e^{-X} = [h_0, h_0] - [s_0^- e^{-X},\ s_0^+ e^{-X}] = \\ &= [(h_0 - s_0^+ e^{-X}),\ (h_0 - s_0^- e^{-X})] = \\ &= [\{h_0 - (m_1 + r(1-\alpha))e^{-X}\}, \\ &\quad \{h_0 - (m_1 - r(1-\alpha))e^{-X}\} = [h(\alpha, X)^-, h(\alpha, X)]. \end{aligned} \tag{12}$$

The functions $h(\alpha, X)^-$ and $h(\alpha, X)^+$ are both differentiable w.r.t. x and we have:

$$\begin{aligned} h'_x(\alpha, X)^- &= (m_1 + r(1-\alpha))e^{-X},\ h'_x(\alpha, X)^+ = \\ &= (m_1 - r(1-\alpha))e^{-X} \end{aligned} \quad , \tag{13}$$

where

$$(m_1 + r(1-\alpha))e^{-X} \geq (m_1 - r(1-\alpha))e^{-X}, \text{ or}$$
$$h'_x(\alpha, X)^- \geq h'_x(\alpha, X)^+.$$

According to Lemma 1 the gH-derivative of $\tilde{h}$ is:

$$\begin{aligned} [\tilde{h}]'_x\big|^\alpha &= \{\min[h'_x(\alpha, X)^-,\ h'_x(\alpha, X)^+], \\ &\quad \max[h'_x(\alpha, X)^-,\ h'_x(\alpha, X)^+]\} = \\ &= [\min\{(m_1 + r(1-\alpha))e^{-X},\ (m_1 - r(1-\alpha))e^{-X}\}, \\ &\quad \max\{(m_1 + r(1-\alpha))e^{-X},\ (m_1 - r(1-\alpha))e^{-X}\}] = \\ &= \{[\tilde{h}'(x)]_\alpha^-, [\tilde{h}'(x)]_\alpha^+\}, \end{aligned}$$

providing that the two functions:

$$\tilde{h}'(x)\big|_\alpha^- = \min[h'_x(\alpha,X)^-,\ h'_x(\alpha,X)^+]$$ and

$$\tilde{h}'(x)\big|_\alpha^+ = \max[h'_x(\alpha,X)^-,\ h'_x(\alpha,X)^+]$$,

define a fuzzy number with respect to α.

We have now:

$$h'(x)\big|_\alpha^- = \min[h'_x(\alpha,X)^-,\ h'_x(\alpha,X)^+] = \\ = m_1 - r(1-\alpha)e^{-X},$$

$$h'(x)\big|_\alpha^+ = \max[h'_x(\alpha,X)^-,\ h'_x(\alpha,X)^+] = \\ = m_1 + r(1-\alpha)e^{-X}$$

$$\frac{\partial h'_x(\alpha,X)^-}{\partial\alpha} = +re^{-X},\quad \frac{\partial h'_x(\alpha,X)^+}{\partial\alpha} = -re^{-X}$$,

$$\tilde{h}'(x)\big|_1^- = \tilde{h}'(x)\big|_1^+ .$$

So the two functions $\tilde{h}'(x)\big|_\alpha^-$, $\tilde{h}'(x)\big|_\alpha^+$, define a fuzzy number with respect to α.

For the above relations we have:

$$h'_x(1,X)^- = m_1 = h'_x(1,X)^+ = m_1$$,

$$(\delta h_\alpha^-)'_x(x) = \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x}f_\alpha^-(x)\right) = \\ = \frac{\partial\{m_1 + r(1-\alpha)e^{-X}\}}{\partial\alpha} = \\ = -re^{-X} \le 0, \quad \text{for } \forall\alpha\in[0,1], \quad (14a)$$

$$(\delta h_\alpha^+)'_x(x) = \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x}f_\alpha^+(x)\right) = \\ = \frac{\partial\{m_1 - r(1-\alpha)e^{-X}\}}{\partial\alpha} = \\ = +re^{-X} \ge 0, \quad \text{for } \forall\alpha\in[0,1], \quad (14b)$$

According to [3], (14a) and (14b) determine that the function $\tilde{h}(\alpha,X)$ is (ii) gH-differentiable.

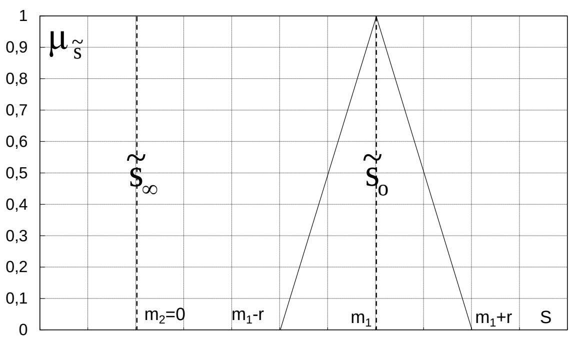

Fig. 2. Boundary conditions

E. *Discharge calculation*

The discharge per unit length is given by:

$$q = KB\frac{dh}{dx} = \frac{KB}{\lambda}\frac{dh}{dX}. \quad (15)$$

The derivative dh/dX is taken from (13) and (15) takes now the following form in fuzzy notation:

$$\tilde{q}\big|^\alpha = \frac{KB}{\lambda}[(m_1 - r(1-\alpha))e^{-X},\ (m_1 + r(1-\alpha))e^{-X}]$$

III. APPLICAIONS

A. *Application 1*

It is assumed that the aquifer has the following characteristics:
K=0.4cm/s (gravel), K′=0.01cm/s (sand), h1=B=20m, h0=24m, h1=20m, B′′=0.8m, λ=√(KBB′/K′)= 25.29m, KB/λ=0.003162m/s, m1=4m, r=1m.
The solution is:

$$\tilde{h}\big|^\alpha = [\{h_0 - (m_1 + r(1-\alpha))e^{-X}\}, \\ \{h_0 - (m_1 - r(1-\alpha))e^{-X}\}] = \\ = [\{24 - (4 + (1-\alpha))e^{-X}\}, \\ \{24 - (4 - (1-\alpha))e^{-X}\}]$$

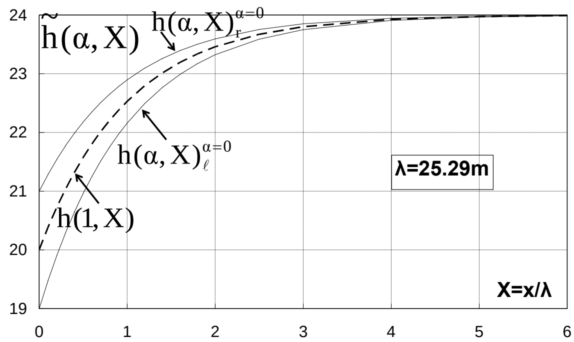

Fig.3 Variation of solution $\tilde{h}(\alpha,X)$ as a function of distance X.

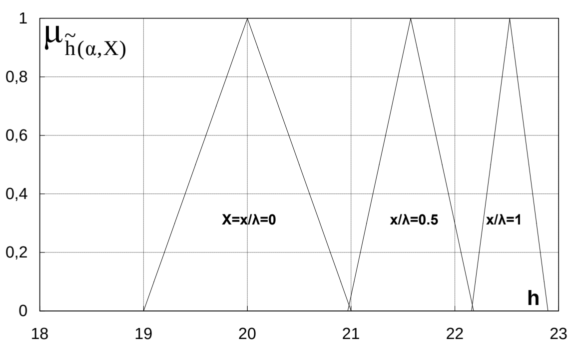

Fig.4 Membership function of the solution at x/λ=0, 1, 2.

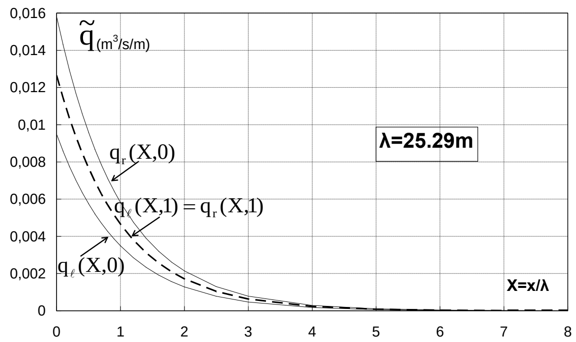

Fig.5 Variation of $\tilde{q}$ as a function of X.

In position (X=0) the discharge towards the Lake is:

$$\tilde{q}\Big|^{X=0} = \frac{KB}{\lambda}[(m_1 - r(1-\alpha))e^{-X=0},\ (m_1 + r(1-\alpha))e^{-X=0}] = $$
$$= 0.003162\,[(\alpha+3),(5-\alpha)]$$

This discharge for α=0, takes the following values:

$$\tilde{q}^{\alpha=0}_{X=0} = [0.0095\,,\ 0.0158\,]m^3s^{-1}m^{-1}.$$

Assuming that the dam has a crest length100 m, the discharge towards the Lake takes the following value:

$\tilde{q}^{\alpha=0}_{X=0} = [0.948\,,1.581\,]m^3s^{-1}$, and for time span of one day the discharge volume will be:

$$\tilde{V}^{\alpha=0}_{X=0} = [81\,966,136\,610]m^3/d,$$

the volume for α=0.5 takes the following value :

$$\tilde{V}^{\alpha=0.5}_{X=0} = [95\,627,122\,949]m^3/d.$$

and finally for a=1 the discharge volume will be:

$$\tilde{V}^{\alpha=1}_{X=0} = [109\,288,109\,288]m^3/d.$$

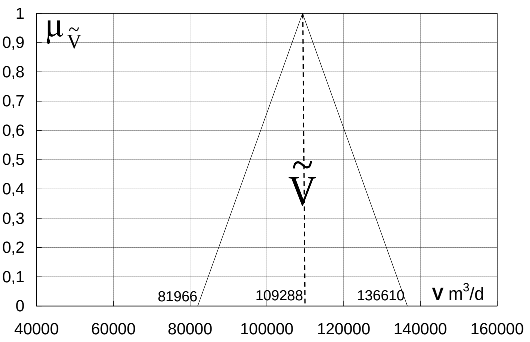

Fig.6 Discharge volume $\tilde{V}$ towards the Lake.

B. Application 2

We assume an aquifer with the following characteristics: K=0.2cm/s(gravel), $K' = 0.04$ cm/s(sand), B=16m, $B' = 3$ m, λ=15.49m, m1=6m, r=1m, s(0)=6m, KB/λ=0.002066m/s, h0=22m, h1=16m.

The solution is:

$$\tilde{h}\Big|^{\alpha} = [\{h_0 - (m_1 + r(1-\alpha))e^{-X}\},$$
$$\{h_0 - (m_1 - r(1-\alpha))e^{-X}\}] =$$
$$= [\{22-(6+(1-\alpha))e^{-X}\},$$
$$\{22-(6-(1-\alpha))e^{-X}\}]$$

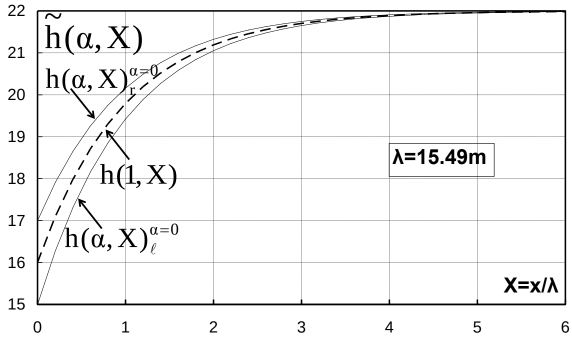

Fig.7. Variation of $\tilde{h}(\alpha,X)$ as a function of distance X.

In position (X=0), the discharge towards the Lake is:

$$\tilde{q}\Big|^{X=0} = \frac{KB}{\lambda}[(m_1 - r(1-\alpha))e^{-X=0},$$
$$(m_1 + r(1-\alpha))e^{-X=0}\,] =$$
$$= 0.002066\,[(\alpha+5),(7-\alpha)].$$

This discharge for α=0, takes the following value:

$$\tilde{q}^{\alpha=0}_{X=0} = [0.01033\,,\ 0.01446\,]m^3s^{-1}m^{-1}.$$

Assuming again a dam with 100m crest length:

$$\tilde{q}^{\alpha=0}_{X=0} = [1.033,\ 1.446\,]m^3s^{-1},$$

and during a day the discharge will be:

$$\tilde{V}^{\alpha=0}_{X=0} = [89251,124951]m^3/d.$$

the volume for α=0.5 takes the following values:

$$\tilde{V}^{\alpha=0.5}_{X=0} = [98157,116003]m^3/d.$$

and finally for α=1 the discharge becomes:

$$\tilde{V}^{\alpha=1}_{X=0} = [107101,107101]m^3/d.$$

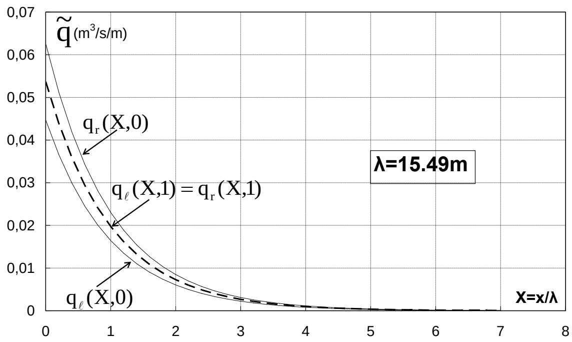

Fig.8 Variation of $\tilde{q}$ as a function of X.

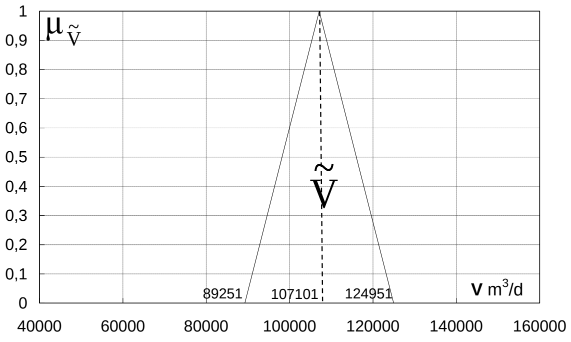

Fig.9 Discharge volume $\tilde{V}$ towards the Lake.

C. Application 3

We assume an aquifer with the following characteristics:

K=0.2cm/s(gravel), K′=0.01cm/s(sand), h1=B=20m, h0=24m, B′′=0.8m, λ=√(KBB′/K′)= 17.88m, KB/λ=0.002236m/s, m1=4m, r=1m.

The characteristics are the same as in the example 1, except the hydraulic conductivity K, which is the half of the value of example 1. For that reason the values of λ and KB/λ have changed. The variation of $\tilde{h}(\alpha, X)$ vs. X, remains the same as in Fig. 1, with different λ, but the variation of $\tilde{q}$ vs. X, as well as the value of discharge volume $\tilde{V}$ towards the Lake have completely changed, because of the change of λ and the KB/λ. The fig. 10, 11, 12 show the existing differences between the example 1 and example 3.

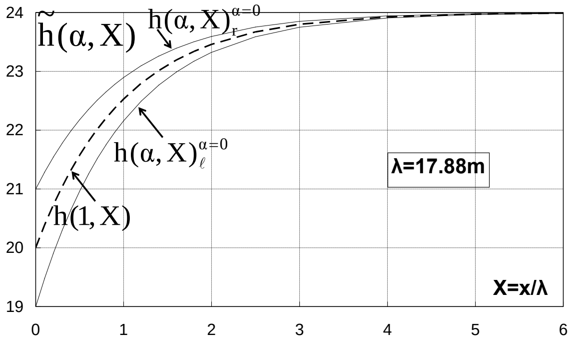

Fig.10 Variation of $\tilde{h}(\alpha, X)$ as a function of distance X.

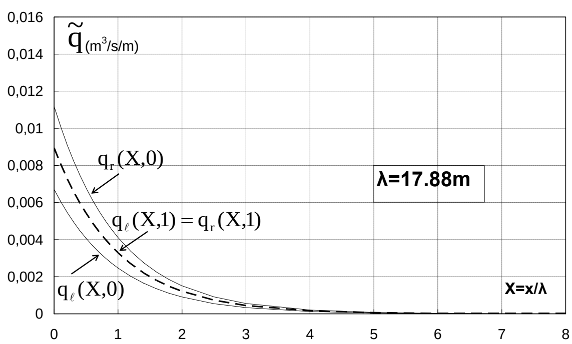

Fig.11 Variation of $\tilde{q}$ as a function of X.

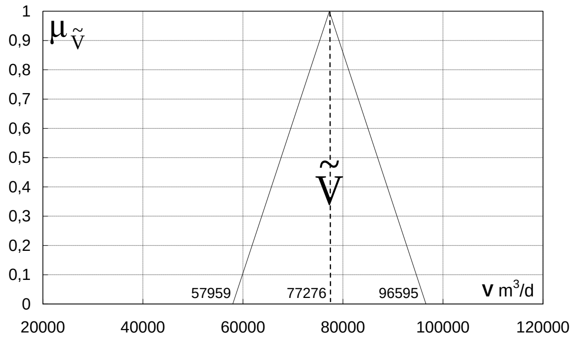

Fig.12 Discharge volume $\tilde{V}$ towards the Lake.

IV. COMMENTS-CONCLUSION

The function $\tilde{h}(\alpha, X)$ is (ii) gH-differentiable and its derivative $\tilde{h}'_x(\alpha, X)$ equals:

$$[\tilde{h}]'_x\Big|^{\alpha} = \{\min[\, h'_x(\alpha, X)^-,\ h'_x(\alpha, X)^+],$$
$$\max[\, h'_x(\alpha, X)^-,\ h'_x(\alpha, X)^+]\}$$

and so a solution to this problem can be obtained.

The fuzziness is decreased with increasing X=x/λ and practically for X > 4 in both cases the values of h and q became crisp numbers.

As it is shown in the above examples the discharge volume decreases as α, which is called the confidence level, is increased. Consequently the interval $[V_r^\alpha, V_\ell^\alpha]$ is decreased by 50% accordingly to the increase of α.

The discharge volume in the second case is decreased by a percentage of 2% compared to the volume in the first, even though the hydraulic conductivity is reduced by 50% and the thickness of the aquifer is reduced by 20%. This is due to the fact that the value of KB/λ in second case multiplied by the slope of the head at x=0, provides a value which is only 2% less than in the first case.

The discharge volume in the third case is decreased by a percentage of 30% compared to the volume in the first, even though the hydraulic conductivity is reduced by 50%. This is due to the fact that the value of leakage factor λ in third case is reduced by 30%, as well as the value of KB/λ, and $\tilde{q}$ is multiplied by this value.

So in our case the leakage factor and the slope of the head function have a very significant role in the solution.

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