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On Similar Partner Curves in Bishop Frames with Variable Transformations

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Abstract—**In this paper, we define a new family of curves and call it a family of similar curves with variable transformation according to the Bishop Frames . Also we give some characterizations of this family and we give some theorems. We obtain that similar curves with variable transformation with vanishing curvatures form the families of similar curves with variable transformation according to the Bishop Frames in $E^3$and E^4.**

Index Terms—**Regular curves, Bishop frame, similar curve, variable transformation.**

I. Introduction

The curves are a part of our lives are the indispensable. For example, heart chest film with X-ray curve, how to act is important to us. Curves give the movements of the particle in Physics.

Helical curves are very important type of curves. Because, helices are among the simplest objects in the art, molecular structures, nature, etc. For example, the path, arroused by the climbing of beans and the orbit where the progressing of the screw are a helix curves. Also, in medicine DNA molecule is formed as two intertwined helices and many proteins have helical structures, known as alpha helices. So, such curves are very important for understand to nature. Therefore, lots of author interested in the helices and they published many papers in Euclidean 3 and 4 - space (See for details: [1] [2]).

Helix curve is defined by the property that the tangent vector field makes a constant angle with a fixed direction. In 1802, M. A. Lancert first proposed a theorem and in 1845, B. de Saint Venant first proved this theorem: "A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant" [7].

Recently, many studies have been reported on generalized helices and inclined curves (Generalized helix is called as inclined curve in n - dimensional Euclidean space ^{n} , $n \geq 4$) [1], [3],[6]. The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. Curvature of the curve may vanish at some points on the curve, that is,second derivative of the curve may be zero. In this situation, we need an alternative frame in E^3 Therefore in [8], Bishop defined a new frame for a curve and called it Bishop frame which is well defined even when the curve has vanishing second derivative in 3- dimensional Euclidean space ^{3} . Similarly, Gökçelik et al. defined a new frame for a curve and they called it parallel transport frame in E^4 [5]. The parallel transport frame is an alternative frame defined a moving frame. In [5], they consider a regular curve α(s) parametrized by s and they defined a normal vector field V(s) which is perpendicular to the tangent vector field $T(s)$ of curve $\alpha(s)$ said to be relatively parallel vector field if its derivative is tangential along the curve $\alpha(s)$. They use the tangent vector $T(s)$ and three relatively parallel vector fields to construct this alternative frame. They choose any convenient arbitrary basis $\{M_1(s), M_2(s), M_3(s)\}$ of the frame, which are perpendicular to T(s) at each point. The derivatives of $\{M_1(s), M_2(s), M_3(s)\}$ only depend on (s) . It is called as parallel transport frame along a curve because the normal component of the derivatives of the normal vector field is zero. The advantages of the parallel frame and the comparable parallel frame with the Frenet frame in 3-dimensional Euclidean space E^3 was given and studied by Bishop [8].

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In this paper, we define similar curves with variable transformation according to the Bishop Frame in Euclidean space E^3 and give some characterizations of these curves. We hope the results of this characterizations will be helpful to mathematicians who are specialized on mathematical modeling as well as other applications of interest.

II. Preliminaries

Let $\gamma: I \subset R \to E^3$ be arbitrary curve in the Euclidean space E^3. γ is said to be of unit speed (or parametrized by arc-length function s) if $\|\gamma'(s)\| = 1$. Then the derivatives of the Frenet frame of γ (Frenet-Serret formula);

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (1)$$

where $\{T, N, B\}$ is the Frenet frame of γ and K, k are the curvature and torsion of curve , respectively [10].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well-defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. Therefore, the Bishop (frame) formulas are expressed as

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}$$

where $\{T, M_1, M_2\}$ is the Bishop Frame and k_1, k_2 are called first and second Bishop curvatures, respectively [8]. The relation between Frenet frame and Bishop frame is given as follows:

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}$$

where θ(s)=arctan (k_2)/(k_1), τ(s)=dθ(s)/ds and $K(s) = \sqrt{k_1{}^2(s) + k_2{}^2(s)}$. Here Bishop curvatures are defined by $k_1 = Kcos\theta$, $k_2 = Ksin\theta$. Where K, k denote principal curvature functions according to Frenet frame of the curve γ [9]. We can parellel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The derivatives of $\{M_1(s), M_2(s)\}$ only depend on $T(s)$. Here the set $\{T(s), M_1(s), M_2(s)\}$ is called as parallel transport frame and $k_1(s) = \langle T'(s), M_1(s) \rangle, k_2(s) = \langle T'(s), M_2(s) \rangle$, called as parallel transport curvatures of the curve γ.

Let $w_\alpha^{(4)} = w_\alpha^{(4)}(s): I \to E^4$ be an arbitrary curve in the four dimesional Euclidean space E^4. Recall that the curve $w_\alpha^{(4)}$ is said to be of unit speed (or parameterized by arc-length function s) if

$$\langle (w_\alpha^{(4)})'(s), (w_\alpha^{(4)})'(s) \rangle = 1,$$

where $\langle , \rangle$ is the standard inner product of E^4 given by

$$\langle X, X \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$. In particular, the norm of a vector $X \in E^4$ is given by $\|X\|^2 = \langle X, X \rangle$. Let {T,N,$B_1$,$B_2$} be the Frenet frame along the unit speed curve $w_\alpha^{(4)}$. Then T, N, B_1 and B_2 are the tangent, the principal normal, first and second binormal vectors of the curve $w_\alpha^{(4)}$, respectively. If $w_\alpha^{(4)}$ is a space curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$\begin{aligned} T'(s) &= K\,N(s) \\ N'(s) &= -K\,T(s) + k\,B_1(s) \\ B_1'(s) &= -k\,N(s) + \tau\,B_2(s) \\ B_2'(s) &= -\tau\,B_1 \end{aligned}$$

where K, k and τ denote principal curvature functions according to Frenet frame of the curve $w_\alpha^{(4)}$ [9]. The parallel transport frame is an alternative frame defined a moving frame. Curvature of the curve may vanish at some points on the curve, that is, *the i-th* $(1 < i < 4)$ derivative of the curve may be zero. We can parellel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The derivatives of $\{M_1(s), M_2(s), M_3(s)\}$ only depend on (s) . Here the set $\{T(s), M_1(s), M_2(s), M_3(s)\}$ is called as parallel transport frame and $k_1(s) = \langle T'(s), M_1(s) \rangle$, $k_2(s) = \langle T'(s), M_2(s) \rangle$, $k_3(s) = \langle T'(s), M_3(s) \rangle$ called as parallel transport curvatures of the curve $w_\alpha^{(4)}$.

Theorem. The alternative paralel transport frame equations are given by

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

where k_1, k_2, k_3 are principal curvature functions according to paralel transport frame of the curve $w_\alpha^{(4)}$ and their expression as follows:

$$\begin{aligned} k_1 &= K.cos\theta cos\varphi \\ k_2 &= K\,(-cos\varphi sin\phi + sin\theta cos\varphi) \\ k_3 &= K\,(sin\phi sin\varphi + cos\phi sin\theta cos\varphi) \end{aligned}$$

$$\begin{aligned} K(s) &= \sqrt{k_1{}^2 + k_2{}^2 + k_3{}^2}, \\ k(s) &= -\varphi' + \phi' sin\theta, \\ \tau(s) &= \theta'/sin\varphi \end{aligned}$$

where K, k and τ are principal curvature functions according to Frenet frame of the curve $w_\alpha^{(4)}$ and θ, ϕ, φ are angles between vectors of parallel transport frame .

III. ON SIMILAR PARTNER CURVES IN BISHOP FRAMES E^3

Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ be curves in the three-dimensional Euclidean space E^3 parameterized by arclengths s_α and s_β with non-zero curvatures $\{k_{1\alpha}, k_{2\alpha}\}$, $\{k_{1\beta}, k_{2\beta}\}$ and Bishop Frames $\{T^\alpha, M_1^\alpha, M_2^\alpha\}$, $\{T^\beta, M_1^\beta, M_2^\beta\}$, respectively. $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ are called similar curves with variable transformation λ_α^β if there exists a variable transformation

$$s_\alpha = \int \lambda_\alpha^\beta(s_\beta) ds_\beta$$

of the arc-length s such that the tangent vectors are the same for the two curves i.e.,

$$T^\beta = T^\alpha \qquad (2)$$

for all corresponding values of parameters under the transformation λ_α^β. Where λ_β^α is arbitrary function of the arclength [4], [9]. It is worth nothing that $\lambda_\alpha^\beta \lambda_\beta^\alpha = 1$. All curves satisfying equation (2) are called a family of similar curves with variable transformations. If we integrate the equality (2) we obtain the following theorem:

Theorem. The position vectors of the family similar curves with variable transformation according to Bishop frame in E^3 can be written in the following form,

$$\gamma_\beta(s_\beta) = \int T^\alpha\left(s_\alpha(s_\beta)\right) ds_\beta = \int T^\alpha(s_\alpha)\lambda_\alpha^\beta ds_\beta.$$

Theorem. Let $\gamma = \gamma(s)$ be an unit speed curve parameterized by arc-length s according to Bishop frame in E^3. Suppose that $\gamma = \gamma(\phi)$ be another parametrization of the curve with parameter $\phi = \int k_1(s)ds$. Then the tangent vector T of $\gamma(s)$ satisfies a vector differential equation of third order given by

$$[\frac{1}{f'}[T'' + (1+f^2)T]]' + f\,T = 0, \qquad (3)$$

where $f(\varphi) = k_2(\varphi)/k_1(\varphi)$, $(T)' = (dT)/d\varphi$,

$$(T)'' = (d^2T)/d\varphi^2.$$

Proof. If we write derivatives given in (1) according to ϕ, We have

$$\frac{dT}{d\varphi} = (k_1M_1 + k_2M_2)\frac{1}{k_1} = M_1 + \frac{k_2}{k_1}M_2$$

$$\frac{dM_1}{d\varphi} = -k_1T = -T$$

$$\frac{dM_2}{d\varphi} = -k_2T = -f\,T$$

respectively, where $f(\varphi) = k_2(\varphi)/k_1(\varphi)$. Then corresponding matrix form of (1) can be given

$$\left[\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 & f \\ -1 & 0 & 0 \\ -f & 0 & 0 \end{bmatrix}\left[\begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}\right] \qquad (4)$$

If we substitute the first equation of new Frenet equation (4) to second and third of (4), we have a vector differential equation of third order (3) as desired.

Theorem. Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ be curves . Then $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ are similar curves with variable transformation according to Bishop frame in the three-dimensional Euclidean space E^3 if and only if the principal normal and binormal vectors are the same for all curves

$$M_1^\beta(s_\beta) = M_1^\alpha(s_\alpha) \qquad (5)$$

$$M_2^\beta(s_\beta) = M_2^\alpha(s_\alpha) \qquad (6)$$

under the particular variable transformation

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{k_{1\alpha}}{k_{1\beta}} \qquad (7)$$

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{k_{2\alpha}}{k_{2\beta}} \qquad (8)$$

of the arc-lengths.

Proof. Let Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ are similar curves with variable transformation. Then differentiating the equality $T^\beta = T^\alpha$
with respect to (s_β) we have

$$\begin{Bmatrix} k_{1\beta}(s_\beta)M_1^\beta(s_\beta) + k_{2\beta}(s_\beta)M_2^\beta(s_\beta) \\ = [k_{1\alpha}(s_\alpha)M_1^\alpha(s_\alpha) + k_{2\alpha}(s_\alpha)M_2^\alpha(s_\alpha)]\dfrac{ds_\alpha}{ds_\beta} \end{Bmatrix} \qquad (9)$$

where

$$\begin{Bmatrix} k_{1\beta}(s_\beta)M_1^\beta(s_\beta) = k_{1\alpha}(s_\alpha)M_1^\alpha(s_\alpha)\dfrac{ds_\alpha}{ds_\beta} \\ k_{2\beta}(s_\beta)M_2^\beta(s_\beta) = k_{2\alpha}(s_\alpha)M_2^\alpha(s_\alpha)\dfrac{ds_\alpha}{ds_\beta}. \end{Bmatrix}$$

(⇐:) Let Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ be curves in the three-dimensional Euclidean E^3 satisfying the equations (5) and (6). If we multiply equation $(\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{k_{1\alpha}}{k_{1\beta}})$ by $k_{1\beta}(s_\beta)$, equation (3.8) by $k_{2\beta}(s_\beta)$ and integrate the result with respect to s_β we have

$$\int k_{1\beta}(s_\beta)M_1^\beta(s_\beta) = \int k_{1\alpha}(s_\alpha)M_1^\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta}$$

$$\int k_{2\beta}(s_\beta)M_2^\beta(s_\beta) = \int k_{2\alpha}(s_\alpha)M_2^\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta}. \qquad (10)$$

From the equation (5), (6), (7) and (8), equation (9) takes the form

$$\int k_{1\beta}(s_\beta)M_1^\beta(s_\beta) + k_{2\beta}(s_\beta)M_2^\beta(s_\beta)$$

$$= \int [k_{1\alpha}(s_\alpha)M_1^\alpha(s_\alpha) + k_{2\alpha}(s_\alpha)M_2^\alpha(s_\alpha)]\frac{ds_\alpha}{ds_\beta}$$

which leads to (2) and the proof is complete.

Theorem. Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ be curves in the three-dimensional Euclidean E^3 . Then $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ are similar curves with variable transformation if and only if ratios of k_1, k_2, k_3 curvatures are the same for all curves

$$\frac{k_{2\beta}(s_\beta)}{k_{1\beta}(s_\beta)} = \frac{k_{2\alpha}(s_\alpha)}{k_{1\alpha}(s_\alpha)} \qquad (11)$$

under the particular variable transformations

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{k_{1\alpha}}{k_{1\beta}} = \frac{k_{2\alpha}}{k_{2\beta}}$$

keeping equal total curvatures, i.e.,

$$\varphi(s_\beta) = \int k_{1\beta}(s_\beta)\, ds_\beta = \int k_{1\alpha}(s_\alpha) ds_\alpha = \varphi\,(s_\alpha) \quad (12)$$

of the arc-lengths.

Proof. ⇒) Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ be regular curves in the three- dimensional Euclidean E^3 . Then there exists a variable transformation of the arc-length s such that the normal and the binormal vectors are the same. Differentiating the equations $M_1^\beta(s_\beta) = M_1^\alpha(s_\alpha)$

and $M_2^\beta(s_\beta) = M_2^\alpha(s_\alpha)$ we have

$$-k_{1\beta}(s_\beta)T^\beta(s_\beta) = -k_{1\alpha}(s_\alpha)T^\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta}$$

$$-k_{2\beta}(s_\beta)M_2^\beta(s_\beta) = -k_{2\alpha}(s_\alpha)M_2^\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta}$$

which leads to the following two equations

$$\begin{Bmatrix} k_{1\beta}(s_\beta) = k_{1\alpha}(s_\alpha)\frac{ds_\alpha}{ds_\beta} & (13) \\ k_{2\beta}(s_\beta) = k_{2\alpha}(s_\alpha)\frac{ds_\alpha}{ds_\beta} & (14) \end{Bmatrix}$$

The variable transformation (11) is the equation (13) after integration. Dividing the above two equations (13) and (14), we obtain the equation (11) under the variable transformations (12).

(⇐:) Let $\gamma_\alpha(s_\alpha)$ and $\gamma_\beta(s_\beta)$ be curves such that the equation (11) is satisfied under the variable transformation (12) of the arclengths. From theorem 3, the tangent vectors $T^\beta(s_\beta)$ And $T^\alpha(s_\alpha)$ of the two curves satisfy vector differential equations of *third order* as follows:

$$\begin{Bmatrix} [\frac{1}{f_\alpha'(\theta_\alpha)}[(T^\alpha(\theta_\alpha))'' + \\ (1+f_\alpha(\theta_\alpha))^2(T^\alpha(\theta_\alpha)]]' + f_\alpha T^\alpha(\theta_\alpha) = 0, \end{Bmatrix} \quad (15)$$

$$\begin{Bmatrix} [\frac{1}{f_\beta'(\theta_\beta)}[(T^\beta(\theta_\beta))'' + \\ (1+f_\beta(\theta_\beta))^2(T^\beta(\theta_\beta)]]' + f_\beta T^\beta(\theta_\beta) = 0, \end{Bmatrix} \quad (16)$$

Where

$$f_\alpha(\theta_\alpha) = \frac{k_{2\alpha}(\theta_\alpha)}{k_{1\alpha}(\theta_\alpha)},\ f_\beta(\theta_\beta) = \frac{k_{2\beta}(\theta_\beta)}{k_{1\beta}(\theta_\beta)}$$

The equation (11) causes

$$f_\alpha(\theta_\alpha) = f_\beta(\theta_\beta)$$

under the variable transformations $\theta_{\alpha=}\theta_\beta$. So that the two equation (15) and (16) under the equation (11) and the transformation (12) are the same. Hence the solution is the same, i.e., the tangent vectors are the same which completes the proof of the theorem.

IV. ON SIMILAR PARTNER CURVES IN BISHOP FRAMES E^4

Let $w_\alpha^{(4)} = w_\alpha^{(4)}(s): I \to E^4$ and $w_\beta^{(4)} = w_\beta^{(4)}(s): I \to E^4$ be curves in the four-dimensional Euclidean space E^4 with arclengths s_α and s_β with non-zero curvatures $\{k_{1\alpha}, k_{2\alpha}, k_{3\alpha}\}$, $\{k_{1\beta}, k_{2\beta}, k_{3\beta}\}$ and Bishop frames $\{T^\alpha, M_1^\alpha, M_2^\alpha, M_3^\alpha\}$, $\{T^\beta, M_1^\beta, M_2^\beta, M_3^\beta\}$, , respectively. $w_\alpha^{(4)}$ and $w_\beta^{(4)}$ are called similar curves with variable transformation λ_α^β if there exists a variable transformation[4],

$$s_\alpha = \int \lambda_\alpha^\beta(s_\beta) ds_\beta$$

of the arc-lengths such that the tangent vectors are the same for the two curves i.e.,

$$T^\beta(s_\beta) = T^\alpha(s_\alpha)$$

for all corresponding values of parameters under the transformation λ_α^β.

Theorem. Let $w^{(4)}(s)$ be a curve in the four-dimensional Euclidean space E^4 parameterized by arc-length s . Provided that $w^{(4)}(\varphi)$ be another parametrization of the curve with parameter $\varphi = \int k_1(s)ds$. Then the unit tangent vector T of $w^{(4)}(s)$ satisfies a vector differential equation of fourth order as follows:

$$\begin{Bmatrix} \left[\frac{1}{f'}\left[T'' + (1+f^2+g^2)T\right]\right]'' \\ +[f + \left(\frac{g'}{f'}\right)]T = 0 \end{Bmatrix} \quad (17)$$

where $f(\varphi) = \frac{k_2(\varphi)}{k_1(\varphi)}$, $g(\varphi) = \frac{k_3(\varphi)}{k_1(\varphi)}$,

$$(T)' = \frac{dT}{d\varphi}, \qquad (T)'' = \frac{d^2T}{d\varphi}$$

Proof.
$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

In the above matrix ,If we write derivaties according to φ, We obtaine

$$\begin{cases} \frac{dT}{d\varphi} = M_1 + \frac{k_2}{k_1} M_2 + \frac{k_3}{k_1} M_3 \\ \frac{dM_1}{d\varphi} = -k_1 T = -T \\ \frac{dM_2}{d\varphi} = -k_2 T = -f\,T \\ \frac{dM_3}{d\varphi} = -k_3 T = -g\,T \end{cases} \quad (18)$$

respectively, where
$f(\varphi) = k_2(\varphi)/k_1(\varphi)$, $g(\varphi) = k_3(\varphi)/k_1(\varphi)$. Then corresponding matrix form is

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & f & g \\ -1 & 0 & 0 & 0 \\ -f & 0 & 0 & 0 \\ -g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

Theorem. Let $w_\alpha^{(4)}(s_\alpha)$ and $w_\beta^{(4)}(s_\beta)$ be curves . Then $w_\alpha^{(4)}(s_\alpha)$ and $w_\beta^{(4)}(s_\beta)$ are similar curves with variable transformation according to Bishop frame in the four-dimensional Euclidean space E^4 if and only if

$$\begin{cases} M_1^\beta(s_\beta) = M_1^\alpha(s_\alpha) \\ M_2^\beta(s_\beta) = M_2^\alpha(s_\alpha) \\ M_3^\beta(s_\beta) = M_3^\alpha(s_\alpha) \end{cases} \quad (19)$$

vectors are the same for all curves according to Bishop Frame. Under the particular variable transformation

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{k_{1\alpha}}{k_{1\beta}} = \frac{k_{2\alpha}}{k_{2\beta}} = \frac{k_{3\alpha}}{k_{3\beta}} \quad (20)$$

of arc-lenths.

Proof. Let $w_\alpha^{(4)}(s_\alpha)$ and $w_\beta^{(4)}(s_\beta)$ are similar curves with variable transformation. Then differentiating the equality $T^\beta(s_\beta) = T^\alpha(s_\alpha)$ with respect to s_β it follows,

$$\begin{Bmatrix} k_{1\beta}(s_\beta) M_1^\beta(s_\beta) + k_{2\beta}(s_\beta) M_2^\beta(s_\beta) \\ +k_{3\beta}(s_\beta) M_3^\beta(s_\beta) = \{k_{1\alpha}(s_\alpha) M_1^\alpha(s_{\beta\alpha}) \\ +k_{2\alpha}(s_\alpha) M_2^\alpha(s_\alpha) + k_{3\alpha}(s_\alpha) M_3^\alpha(s_\alpha)\} \frac{ds_\alpha}{ds_\beta} \end{Bmatrix}$$

(21)

From (21), we have (19) and (20) immediately.
(⇐:) Let $w_\alpha^{(4)}(s_\alpha)$ and $w_\beta^{(4)}(s_\beta)$ are similar curves with variable transformation satisfying (19) and (20). By multiplying with (19) ; $k_{1\beta}, k_{2\beta}, k_{3\beta}$ } and differentiating the results with respect to s_β we obtain

$$\begin{Bmatrix} -k_{1\beta}(s_\beta) T^\beta(s_\beta) = -k_{1\alpha}(s_\alpha) T^\alpha(s_\alpha) \frac{ds_\alpha}{ds_\beta} \\ -k_{2\beta}(s_\beta) T^\beta(s_\beta) = -k_{2\alpha}(s_\alpha) T^\alpha(s_\alpha) \frac{ds_\alpha}{ds_\beta} \\ -k_{3\beta}(s_\beta) T^\beta(s_\beta) = -k_{3\alpha}(s_\alpha) T^\alpha(s_\alpha) \frac{ds_\alpha}{ds_\beta} \end{Bmatrix}$$

$$\begin{Bmatrix} T^\beta(s_\beta) = \\ \int [k_{1\beta}(s_\beta) M_1^\beta(s_\beta) + k_{2\beta}(s_\beta) M_2^\beta(s_\beta) \\ +k_{3\beta}(s_\beta) M_3^\beta(s_\beta)] ds_\beta \end{Bmatrix}$$

$$\begin{Bmatrix} T^\alpha(s_\alpha) = \\ \int [k_{1\alpha}(s_\alpha) M_1^\alpha(s_\alpha) + k_{2\alpha}(s_\alpha) M_2^\alpha(s_\alpha) \\ +k_{3\alpha}(s_\alpha) M_3^\alpha(s_\alpha)] ds_\alpha \end{Bmatrix}$$

From (19), (20) and (21) we obtain

$$T^\alpha(s_\alpha) = T^\beta(s_\beta)$$

which means that $w_\alpha^{(4)}(s_\alpha)$ and $w_\beta^{(4)}(s_\beta)$ are similarcurves with variable transformation according to Bishop frame in four-dimensional Euclidean space E^4.

Let now consider $w_\alpha^{(4)}(s_\alpha)$ and $w_\beta^{(4)}(s_\beta)$ be curves such that the equation (17) is satisfied under the variable transformation

$$\varphi(s_\beta) = \int k_{1\beta}(s_\beta)\, ds_\beta = \int k_{1\alpha}(s_\alpha) ds_\alpha = \varphi(s_\alpha)$$

of the arclengths. From (17), the tangent vectors $T^\alpha(s_\alpha)$ and $T^\beta(s_\beta)$ of the two curves satisfy vector differential equations of fourty order as follows:

$$\begin{Bmatrix} [\frac{1}{f_\alpha'(\theta_\alpha)} [(T^\alpha(\theta_\alpha))'' + \\ [1 + f_\alpha(\theta_\alpha))^2 + g_\alpha(\theta_\alpha))^2] T^\alpha(\theta_\alpha)]'' \\ +[f_\alpha(\theta_\alpha) + \left(\frac{(g_\alpha(\theta_\alpha))'}{(f_\alpha(\theta_\alpha))'}\right)] T^\alpha(\theta_\alpha) = 0, \end{Bmatrix}$$

$$\begin{Bmatrix} [\frac{1}{f_\beta'(\theta_\beta)} [(T^\beta(\theta_\beta))'' + \\ [1 + f_\beta(\theta_\beta))^2 + g_\beta(\theta_\beta))^2] T^\beta(\theta_\beta)]'' \\ +[f_\beta(\theta_\beta) + \left(\frac{(g_\beta(\theta_\beta))'}{(f_\beta(\theta_\beta))'}\right)] T^\beta = 0, \end{Bmatrix}$$

Where

$$f_\alpha(\theta_\alpha)=\frac{k_{2\alpha}(\theta_\alpha)}{k_{1\alpha}(\theta_\alpha)},\ f_\beta(\theta_\beta)=\frac{k_{2\beta}(\theta_\beta)}{k_{1\beta}(\theta_\beta)}$$

$$f_\alpha(\theta_\alpha)=\ f_\beta(\theta_\beta)$$

under the variable transformations

$$\theta_\alpha=\theta_\beta.$$

It means that the unit tangent vectors are the same which completes the proof of the theorem.

Example. Let us consider, the Euler Spiral $\gamma(s)=(\gamma_1(s),\gamma_2(s),\gamma_3((s))$ of E^3

$$\begin{Bmatrix}\gamma_1(s)=(3/5)\int\sin(s^2+1)ds\\ \gamma_2(s)=(3/5)\int\cos(s^2+1)ds\\ \gamma_3((s)=\frac{4}{5}\int sds\end{Bmatrix}$$

Then the tangent vector $T(S)$ of $\gamma(s)$ satisfies a vector differential equation of third order given by

$$[(\frac{1}{f'})[T''+(1+f^2)T]]'+f\,T=0.$$

Proof. We are calculated this curve's curvature funcytion with help of Mathematica Programme $K=6s/5$ and $k=-8s/5$ The Frenet -Serret frame of the curve $\gamma=\gamma(s)$ may be written by the aid Mathematica Programme as fallows

$$T(s)=((3/5)sin(s^2+1),(3/5)cos(s^2+1),(4/5)),$$
$$N(s)=(cos(s^2+1),-sin(s^2+1),0),$$
$$B(s)=((4/5)sin(s^2+1),cos(s^2+1),-(3/5)).$$

To create a Bishop Frame to find the angle of rotation .γ=γ(s) has the form

$$\theta(s)=-\int kds=\frac{4s^2}{5}.$$

Transformation matrix for the curve

$$\begin{bmatrix}T\\N\\B\end{bmatrix}=\begin{bmatrix}1&0&0\\0&cos\frac{4s^2}{5}&-sin\frac{4s^2}{5}\\0&sin\frac{4s^2}{5}&cos\frac{4s^2}{5}\end{bmatrix}\begin{bmatrix}T\\M_1\\M_2\end{bmatrix}$$

T,M_1,M_2 can be found

$$T(s)=T(s)$$

$$M_1(s)=cos\frac{4s^2}{5}N(s)+sin\frac{4s^2}{5}B(s)$$

$$M_2(s)=-sin\frac{4s^2}{5}N(s)+cos\frac{4s^2}{5}B(s).$$

Even , first curvature function according to Bishop frame of curve. γ=γ(s) is calculated

$$\begin{Bmatrix}k_1(s)=\langle T'(s),M_1(s)\rangle\\ =(6s/5)cos\frac{4s^2}{5},\end{Bmatrix}$$

$$\begin{Bmatrix}k_2(s)=\langle T'(s),M_2(s)\rangle\\ =(6s/5)cos\frac{4s^2}{5}.\end{Bmatrix}$$

If we write derivatives given in (3.3) according to *s* we have

$$\begin{Bmatrix}[\frac{1}{[\tan(4s/5)]'}[T''+(1+[\tan(4s/5)]^2T]]'+\\ \tan(4s/5)T=0.\end{Bmatrix}$$

where, the tangent vector of $\gamma(s)$ satisfies is a vector differential equation of third order.

CONCLUSION

In the three- dimensional Euclidean E^3 and the four-dimensional Euclidean E^4 according to Bishop frame, the similar curves are defined and some properties of these curves are obtained. It is shown that this curves with vanishing curvatures form the families of similar curves.

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