International Journal of New Technology and Research (IJNTR)
ISSN:2454-4116, Volume-2, Issue-2, February 2016 Pages 12-18

Existence of Rayleigh Waves Over The Heterogeneous Prestressed Medium

Inder Singh Gupta , Amit Kumar

***Abstract*— The frequency equation of Rayleigh waves propagating over the free surface of an prestressedorthotropic perfectly elastic, heterogeneous semi-infinite medium with material properties varying as $\lambda = \lambda_0 e^{az}$, $\mu = \mu_0 e^{az}$, $\rho = \rho_0 e^{az}$ and principal components of the initial stress $(S_{11} = S_{11}^0 e^{az}, S_{22} = S_{22}^0 e^{az}, S_{33} = S_{33}^0 e^{az})$ ($a > 0, z$ being the distance from free surface) has been derived using matrix method. Then for a particular case, for unstressed heterogeneous medium, we showed that, there exist single Rayleigh modes which cannot propagate below certain cut-off frequency. It is found that in case $\lambda = 0$, $\beta < c < c_0$ where c denotes phase velocity. β is the constant shear wave velocity of the medium . c_0 is the corresponding Rayleigh wave velocity of homogenous medium of the same Poisson's ratio. On the variation of λ i.e. 1< λ < $\frac{(3+\sqrt{5})}{4}$, graph has been drawn between $\frac{c}{\beta}$ and $\frac{\pi\omega}{\alpha\beta}$.**

***Index Terms*— Rayleigh wave, heterogeneous medium, prestressed medium and dispersion equation, phase velocity.**

I. Introduction

After the poineer work of Lord Rayleigh (1885), the theory of surface waves inhomogeneous an isotropic half spaces has enjoyed remarkable progress in the 1960's and 1970's. Most of the works have been summarized in these books of Ben-Menahem and Singh (1981), Pilant (1979) and Brekhovskikh(1960) for more detailed study. Due to complexity of the problem, the study of propagation of Rayleigh waves in heterogeneous half-space was not enough. It has been studied in heterogeneous medium among others Stoneley (1934), Wilson (1942), Newlands (1950), Dutta (1963), Hook (1961) and Karlson (1963) .

Singh (1965) discussed the propagation of Rayleigh type waves in an axially symmetric heterogeneous layer lying between two half-spaces. The variation of the parameters in the layer is assumed to be of the form $\frac{\lambda}{\lambda_0} = \frac{\mu}{\mu_0} = \frac{1}{1+\alpha z}$, $\frac{\rho}{\rho_0} = \frac{1}{1+\alpha z^2}$ where α is constant and z is the distance measured from one interface into the layer. He also assumed that the vector wave equation for the layer is separable.

Sidhu(1970) has obtained the frequency equation of Rayleigh waves propagating over the free surface of an isotropic, perfectly elastic ,heterogeneous semi-infinite medium with materials properties varying as $\lambda = \lambda_0\ e^{az}$, $\mu = \mu_0\ e^{az}$ $and\ \rho = \rho_0\ e^{az} (a > 0)$.

He also obtained the solution of the frequency equation in closed form in two cases (1) $\lambda = 0$, (2) $\lambda = \mu$.

Most of these studies have ignored the initial stresses present in the media. Infact, Earth is an initially stressed heterogeneous medium .Therefore it is of interest to deal the problems on Earth models where initial stresses and inhomogeneity factors are taken into account .Biot (1965) has shown that an anisotropy is developed due to initial stress present in the homogeneous elastic medium. Frequency equation of Rayleigh waves in prestressed heterogeneous medium is affected not only by initial stresses but by inhomogeneity factors also. This study is very useful for surface waves analysis.

Due to complexity of the problem, the study of the transmission of Rayleigh waves in heterogeneous half-space was not enough. Some researchers have tried to derive the dispersion equation for Rayleigh waves in prestressed heterogeneous medium e.g. Das et al.(1992), Kakar and Kakar(2013) and Abd –Alla et al.(2009).Most of them used potential method to solve these problems .Sidhu and Singh (1983) and Norries(1983) also have pointed out that potential method is not suitable for prestressed media .

Authors wants to discuss the propagation of Rayleigh waves over the free surface of a prestressed heterogeneous half-space in which $\frac{\lambda}{\lambda_0} = e^{az}$, $\frac{\mu}{\mu_0} = e^{az}$ and $\frac{\rho}{\rho_0} = e^{az}$ and components of normal initial stresses ($\frac{S_{11}}{S_{11}^0} = e^{az}$, $\frac{S_{22}}{S_{22}^0} = e^{az}$ and $\frac{S_{33}}{S_{33}^0} = e^{az}$) ,(a >0), z being the distance from free surface .In the present study we investigate the equation of motion with the help of matrix method. This method of obtaining the solution of equations of motion, can be readily applied to obtain the frequency equations of Rayleigh waves in a heterogeneous layer over laying in a heterogeneous half-space .These problem are for further study.

II. Basic Equations

Consider a semi-infinite, perfectly elastic prestressed, heterogeneous medium .The materials is either isotropic in finite strain or anisotropic with orthotropic symmetry .The principal directions of prestress are chosen to coincide with the direction elastic symmetry and the co-ordinate axes .The

Inder Singh Gupta, Department of Mathematics, J.V.M.G.R.R. (P.G.) College, Charkhi Dadri, Bhiwani.
Amit Kumar, Research Scholar, Department of Mathematics, Singhania University, Rajasthan

state of prestress is defined by principal components S_{11}, S_{22} and S_{33} of the initial stress. The general equations of motion for prestressed solid in the absence of external forces is given by Biot (1965).

$$s_{ij,j} + S_{jk}\omega_{ik,j} + S_{ik}\omega_{jk,j} - e_{jk}S_{ik,j} = \rho u_{i,tt}, \qquad (1)$$

where ρ is the density, u_i are the displacement components and , i indicates differentiation with respect to x_i. The incremental stresses s_{ij} are assumed to be linearly related to the incremental strains e_{ij} through the incremental elastic coefficients B_{ij}, Q_1 and Q_2.

$$s_{11} = B_{11}u_x + B_{12}(v_y + w_z),$$
$$s_{22} = (B_{12} - P)u_x + B_{22}v_y + B_{23}w_z,$$
$$s_{33} = (B_{12} - P)u_x + B_{23}v_y + B_{22}w_z, \qquad (2)$$
$$s_{12} = Q_2(u_y + v_x),$$
$$s_{13} = Q_2(u_z + w_x),$$
$$s_{23} = Q_1(w_y + v_z).$$

Where, it is assume that $S_{22} = S_{33}$, S_{11} and S_{33} are constants.

$$(x, y, z) = (x_1, x_2, x_3),$$
$$(u, v, w) = (u_1, u_2, u_3),$$
$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad (3)$$
$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}),$$
$$u_{i,j} = \frac{\partial u_i}{\partial x_j},\ u_{i,t} = \frac{\partial u_i}{\partial t},\ u_x = \frac{\partial u}{\partial x}, \quad \text{etc.}$$

And $P = S_{33} - S_{11}$.

Where

$$B_{11} = (2\mu + \lambda)(1 + \varepsilon_{11} - 2\varepsilon_{22}),$$
$$B_{22} = (2\mu + \lambda)(1 - \varepsilon_{11}), \qquad (4)$$
$$B_{12} = \lambda(1 - \varepsilon_{22}) - S_{11},$$
$$B_{23} = \lambda(1 - \varepsilon_{11}) - S_{33},$$
$$Q_1 = \mu + (\mu + \lambda)\varepsilon_{22} + \frac{1}{2}(\lambda - 2\mu)\varepsilon_{11},$$
$$Q_2 = \mu + \frac{1}{2}(\mu + \lambda)(\varepsilon_{11} + \varepsilon_{22}) + \frac{1}{2}(\lambda - 2\mu)\varepsilon_{22}.$$

Further λ, μ are Lame's constants and ε_{ij} are the incremental strains. s_{ij} and ε_{ij} are related by Hooke's Law.

$$s_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad (5)$$

Here it is assumed that material properties and initial stress components are varying as

$$\lambda = \lambda^0 e^{az}, \mu = \mu^0 e^{az}, \rho = \rho^0 e^{az} \text{ and } S_{11} = S_{11}^0 e^{az}, S_{22} = S_{22}^0 e^{az}, S_{33} = S_{33}^0 e^{az}, P = P^0 e^{az} (a > 0)$$

(6)

Then the incremental elastic coefficients becomes

$$B_{11} = B_{11}^0 e^{az}, B_{22} = B_{22}^0 e^{az}, B_{12} = B_{12}^0 e^{az}, \quad B_{23} = B_{23}^0 e^{az}$$

(7)

$$Q_1 = Q_1^0 e^{az}, Q_2 = Q_2^0 e^{az}, P^0 = S_{33}^0 - S_{11}^0.$$

Where $(B_{11}^0, B_{22}^0, B_{12}^0, B_{23}^0, Q_1^0, Q_2^0)$ and (S_{11}^0, S_{33}^0) are elastic coefficients and initial stresses in homogeneous orthotropic prestressed medium. λ^0, μ^0, ρ^0 are Lame's constants and density of material in unstressed state.

Using (2), (3), (6) and (7) in (1) we get

$$B_{11}^0\frac{\partial^2 u}{\partial x^2} + \left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 w}{\partial x \partial z} + \left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 v}{\partial x \partial y} + \left(Q_2^0 + \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial y^2}$$
$$+ \left(Q_2^0 + \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial z^2} + a\left(Q_2^0 + \frac{P^0}{2}\right)\frac{\partial u}{\partial z} + a\left(Q_2^0 - \frac{P^0}{2}\right)\frac{\partial w}{\partial x} = \rho^0\frac{\partial^2 u}{\partial t^2},$$

$$\left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial x \partial y} + \left(Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 v}{\partial x^2} + B_{22}^0\frac{\partial^2 v}{\partial y^2}$$
$$(B_{23}^0 + Q_1^0)\frac{\partial^2 w}{\partial y \partial z} + Q_1^0\frac{\partial^2 v}{\partial z^2} + aQ_1^0\frac{\partial w}{\partial y} + aQ_1^0\frac{\partial v}{\partial z} = \rho^0\frac{\partial^2 v}{\partial t^2},$$

$$\left(Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 w}{\partial x^2} + \left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial x \partial z} +$$
$$Q_1^0\frac{\partial^2 w}{\partial y^2} + (B_{23}^0 + Q_1^0)\frac{\partial^2 v}{\partial y \partial z} + B_{22}^0\frac{\partial^2 w}{\partial z^2} + a(B_{12}^0 - P^0)$$
$$\frac{\partial u}{\partial x} + aB_{23}^0\frac{\partial v}{\partial y} + aB_{22}^0\frac{\partial w}{\partial z} = \rho^0\frac{\partial^2 w}{\partial t^2}. \qquad (8)$$

III. PLANE SURFACE WAVES SOLUTION

Let

$$u = u_1(z)e^{-i(x\xi + y\eta - pt)},$$
$$v = v_1(z)e^{-i(x\xi + y\eta - pt)}, \qquad (9)$$
$$w = w_1(z)e^{-i(x\xi + y\eta - pt)},$$

where $i = \sqrt{-1}$.

Then displacements in (9) define a plane harmonic wave propagating in the direction of the normal to the plane as shown in Figure 1.

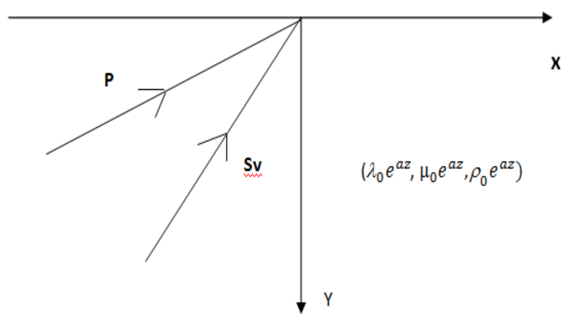

Figure 1

$$x\xi + y\eta = \text{constant}, \qquad (10)$$

with period $\frac{2\pi}{p}$, wave length $\frac{2\pi}{\sigma}$ and phase velocity $c = \frac{p}{\sigma}$ where

$$\sigma^2 = \xi^2 + \eta^2. \qquad (11)$$

Substituting (9) and (11) in (8) and removing the exponential factor , we get

International Journal of New Technology and Research (IJNTR)
ISSN:2454-4116, Volume-2, Issue-2, February 2016 Pages 12-18

$$\mathrm{C}u_1 + A_3\xi\eta v_1 + iaA_2\xi w_1 + i\xi A_3\frac{\partial w_1}{\partial z} - aA_2\frac{\partial u_1}{\partial z} - A_1\frac{\partial^2 u_1}{\partial z^2} = 0,$$

$$A_3\xi\eta u_1 + Dv_1 + iaQ_1^0\eta w_1 + i\eta A_4\frac{\partial w_1}{\partial z} - aQ_1^0\frac{\partial v_1}{\partial z} - Q_1^0\frac{\partial^2 v_1}{\partial z^2} = 0,$$

$$ia(B_{12}^0 - P^0)\xi u_1 + iaB_{23}^0\eta v_1 + Ew_1 + i\xi A_3\frac{\partial u_1}{\partial z} + i\eta A_4\frac{\partial v_1}{\partial z} - al\frac{\partial w_1}{\partial z} - l\frac{\partial^2 w_1}{\partial z^2} = 0,$$

(12)

where

$$C = B_{11}^0\xi^2 + A_1\eta^2 - \rho^0 p^2,$$

$$D = B_{22}^0\eta^2 + A_2\xi^2 - \rho^0 p^2,$$

$$E = Q_1^0\eta^2 + A_2\xi^2 - \rho^0 p^2, A_1 = Q_2^0 + \frac{P^0}{2}, A_2 = Q_2^0 - \frac{P^0}{2},$$

$$A_3 = B_{12}^0 + Q_2^0 - \frac{P^0}{2}, A_4 = B_{23}^0 + Q_1^0, l = B_{22}^0.$$

(13)

The set of (12) can be written in the following matrix form

$$\frac{dT}{dz} = AT. \quad (14)$$

T being the column matrix with elements $T_1, T_2, T_3, T_4, T_5, T_6$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{C}{A_1} & \frac{A_3\xi\eta}{A_1} & \frac{iaA_2\xi}{A_1} & \frac{-aA_2}{A_1} & 0 & \frac{i\xi A_3}{A_1} \\ \frac{A_3\xi\eta}{Q_1^0} & \frac{D}{Q_1^0} & ia\eta & 0 & -a & \frac{i\eta A_4}{Q_1^0} \\ \frac{ia(B^0_{12} - P^0)\xi}{l} & \frac{iaB^0_{23}\eta}{l} & \frac{E}{l} & \frac{iA_3\xi}{l} & \frac{iA_4\eta}{l} & -a \end{pmatrix} \quad (15)$$

where

$$u_1 = T_1, v_1 = T_2, w_1 = T_3, \frac{\partial u_1}{\partial z} = T_4, \frac{\partial v_1}{\partial z} = T_5, \frac{\partial w_1}{\partial z} = T_6.$$

(16)

In order to solve (14), we write

$$T = e^{sz}\vec{C}, \quad (17)$$

$$se^{sz}\vec{C} = Ae^{sz}\vec{C}.$$

Which is satisfied for all z if

$$[A - sI]\vec{C} = 0, \quad (18)$$

where I denoting six order unit matrix.

The set (18) has a non –vanishing solution vector $\vec{C}$ if and only if

$$|A - sI| = 0 \quad (19)$$

The determinant (19) is a sixth order equation in s and gives the following six distinct values,

$$s_1 = \frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) - 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} - a\right],$$

$$s_2 = \frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) + 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} - a\right],$$

$$s_3 = -\frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) - 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} + a\right],$$

(20)

$$s_4 = -\frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) + 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} + a\right],$$

$$s_5 = \frac{1}{2}\left[\sqrt{a^2 + 4p_2} - a\right],$$

$$s_6 = -\frac{1}{2}\left[\sqrt{a^2 + 4p_2} + a\right],$$

where

$$p_1 = \sigma^2 - \frac{p^2}{C_p^2}, p_2 = \sigma^2 - \frac{p^2}{C_s^2}, p_3 = a^2\sigma^2\left(1 - 2\frac{C_s^2}{C_p^2}\right),$$

$$C_p^2 = \frac{B_{22}^0}{\rho_0}, C_s^2 = \frac{Q_2^0 + \frac{P^0}{2}}{\rho_0} \text{ and } \gamma^2 = \frac{Q_2^0 + \frac{P^0}{2}}{B_{22}^0}.$$

(21)

In case the initial state is unstressed, the medium is isotropic and the first order theory of classical elasticity is assumed, then it may be shown that (Biot [10])

$$A_3 = A_4 = \lambda_0 + \mu_0, C_p = \alpha^2, C_s = \beta^2, p_1 = \sigma^2 - \frac{p^2}{\alpha^2}, p_2 = \sigma^2 - \frac{p^2}{\beta^2}, p_3 = a^2\sigma^2(1 - 2\gamma^2),$$

$$B_{11}^0 = B_{22}^0 = \lambda_0 + 2\mu_0, B_{12}^0 = B_{23}^0 = \lambda_0, Q_1^0 = Q_2^0 = \mu_0, A_1 = A_2 = \mu_0,$$

$$\alpha^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \beta^2 = \frac{\mu_0}{\rho_0} \text{ and } \gamma^2 = \left(\frac{\mu_0}{\lambda_0 + 2\mu_0}\right)$$

(22)

α and β being the constant compressional and shear wave velocities in the unstressed homogeneous medium. If the medium becomes homogeneous the putting a=0 in (21) and (20) becomes.

$$s_1 = \sqrt{p_2} = \sigma\sqrt{\left(1 - \frac{c^2}{\beta^2}\right)}, s_2 = \sqrt{p_1} = \sigma\sqrt{\left(1 - \frac{c^2}{\alpha^2}\right)}, s_3 = -\sigma\sqrt{\left(1 - \frac{c^2}{\beta^2}\right)},$$

$$s_4 = -\sigma\sqrt{\left(1 - \frac{c^2}{\alpha^2}\right)}, s_5 = \sigma\sqrt{\left(1 - \frac{c^2}{\beta^2}\right)}, s_6 = -\sigma\sqrt{\left(1 - \frac{c^2}{\beta^2}\right)}. \quad (23)$$

The general solution of (14) is, therefore

$$\vec{T} = \sum_{i=1}^{6} e^{s_i z} C_i \quad (24)$$

Writing the column vector, $\vec{T}$ and $\vec{C_i}$ in the symbolic form as

$\vec{T} = [T_j]$, $\vec{C_i} = [C_{ij}], i, j = (1,2,3, \ldots\ldots, 6)$ in (24) and equating the corresponding elements ,we get

$$u_1 = T_1 = \sum_{i=1}^{6} e^{s_i z} C_{i1},$$

$$v_1 = T_2 = \sum_{i=1}^{6} e^{s_i z} C_{i2}, w_1 = T_3 = \sum_{i=1}^{6} e^{s_i z} C_{i3}.$$

(25)

We replace s by s_i and $\vec{C}$ by $\vec{C_i}$ in (18) to get

$$[A - s_i I][C_j] = 0. \quad (26)$$

For each i ,(26) gives a set of six homogeneous equations in six unknowns $C_{ij}(j = 1,2, \ldots\ldots, 6)$. Since the s_i's have been obtained from (20), we remark that each of the sets of six C_{ij}, for any given i must be consistent .We choose a set of five from these six C_{ij}, i fixed, and express them in term of sixth in such a way that for none of the values of i the coefficient determinant vanishes. This happens if we express $C_{ij}(j = 2, \ldots\ldots, 6)$ in terms of $C_{i1} = K_i(say)$. After some manipulation, we obtain

$$C_{i2} = \begin{Bmatrix} -k_i \frac{\xi}{\eta}, & i = 5,6 \\ k_i \frac{\eta}{\xi}, & i = 1,2,3,4 \end{Bmatrix}, \quad (27)$$

$$C_{i3} = \begin{Bmatrix} 0, & i = 5,6 \\ \frac{i m_i k_i}{\xi_i}, & i = 1,2,3,4 \end{Bmatrix},$$

Where

$$m_i = -\frac{(a\lambda_0 + A_3 s_i)\sigma^2}{(E - l s_i^2 - a l s_i)}, i = 1,2,3,4 \quad (28)$$

and

A_3 is already defined in (13) (29)

Substituting for these in (25), we obtain

$$u_1 = \sum_{i=1}^{6} e^{s_i z} k_i,$$

$$v_1 = \frac{\eta}{\xi}\sum_{i=1}^{4} e^{s_i z} k_i - \frac{\xi}{\eta}\sum_{i=5}^{6} e^{s_i z} k_i, \quad (30)$$

$$w_1 = \frac{i}{\xi}\sum_{i=1}^{4} e^{s_i z} m_i k_i.$$

Appropriate solution to be used from (30) which satisfy the radiation as $z \to \infty$ are for $K_1 = K_2 = K_5 = 0$,where we must ensure that $Re(s_1, s_2, s_3) > 0.$ Therefore putting $K_1 = K_2 = K_5 = 0$ in (30), we get

$$u_1 = \sum_{i=3}^{4} k_i e^{s_i z} + K_6 e^{s_6 z},$$

$$v_1 = \frac{\eta}{\xi}\sum_{i=3}^{4} e^{s_i z} k_i - \frac{\xi}{\eta} e^{s_6 z} k_6, \quad (31)$$

$$w_1 = \frac{i}{\xi}\sum_{i=3}^{4} e^{s_i z} m_i k_i.$$

IV. Boundary conditions

At the free surface boundary conditions are

$\Delta f_x = 0$, $\Delta f_y = 0$, $\Delta f_z = 0$. at z =0. (32)

Here we assume that initial stresses are normal and $S_{22} = S_{33}$ then x, y, z components of the incremental boundary forces may be written explicitly (Biot[1961] as

$$\Delta f_x = (s_{11} + S_{11}e - S_{11}e_{xx})\eta_x + (s_{12} + S_{33}\omega_z - S_{11}e_{xy})\eta_y + (s_{31} + S_{33}\omega_y - S_{11}e_{zx})\eta_z,$$

$$\Delta f_y = (s_{22} + S_{33}e - S_{33}e_{yy})\eta_y + (s_{23} + S_{33}\omega_x - S_{33}e_{yz})\eta_z + (s_{12} + S_{11}\omega_z - S_{33}e_{xy})\eta_x,$$

$$\Delta f_z = (s_{33} + S_{33}e - S_{33}e_{zz})\eta_z + (s_{31} - S_{11}\omega_y - S_{33}e_{zx})\eta_x + (s_{23} + S_{33}\omega_x - S_{33}e_{yz})\eta_y,$$

(33)

when η_j are cosines of the angles between the normal and the jth direction in three –dimensional rectangular Cartesian system. Therefore the values of

$$\Delta f_x = (s_{13} + S_{33}\omega_y - S_{11}e_{zx}) = 0;$$

$$\Delta f_y = (s_{23} - S_{33}\omega_x - S_{33}e_{zy}) = 0; \quad (34)$$

$$\Delta f_z = (s_{33} + S_{33}e - S_{33}e_{zz}) = 0.$$

The above boundary forces, take the form (using (2))

$$\Delta f_x = (Q_2^0 + \frac{P^0}{2})u_z + (Q_2^0 - \frac{R^0}{2})\omega_x = 0,$$

$$\Delta f_y = (Q_1^0 - s_{33}^0)\omega_y + Q_1^0 v_z = 0, \text{ at } z = 0 \quad (35)$$

$$\Delta f_z = (B_{12}^0 + s_{11}^0)u_x +$$

$$(B_{23}^0 + s_{33}^0)v_y + B_{22}^0\omega_z = 0.$$

Where

$R = R^0 (S_{11}^0 + S_{33}^0),$

$\omega_x = -\omega_{23}, \omega_z = -\omega_{12}, \omega_y = -\omega_{31},$

$e = e_{xx} + e_{yy} + e_{zz},$

Using (9),(31) in (35), we get

$$\left[\left(Q_2^0 + \frac{P^0}{2}\right)s_3 + \left(Q_2^0 - \frac{R^0}{2}\right)m_3\right]K_3 + \left[\left(Q_2^0 + \frac{P^0}{2}\right)s_4 + \left(Q_2^0 - \frac{R^0}{2}\right)m_4\right]K_4 + \left(Q_2^0 + \frac{P^0}{2}\right)s_6 K_6 = 0,$$

$$\left[(Q_1^0 - S_{33}^0)m_3 + Q_1^0 s_3\right]K_3 + \left[(Q_1^0 - S_{33}^0)m_4 + Q_1^0 s_4\right]K_4 \frac{\eta}{\xi} - \frac{\xi}{\eta}s_6 K_6 = 0, \quad (36)$$

$$\left[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3\right]K_3 + \left[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4\right]K_4 - (B_{23}^0 - B_{12}^0 + P^0)\xi^2 K_6 = 0$$

Put $K_6 = 0$ in (36) reduce to

$$\left[(Q_2^0 + P^0/2)s_3 + (Q_2^0 - R^0/2)m_3\right]K_3 + \left[(Q_2^0 + P^0/2)s_4 + (Q_2^0 - R^0/2)m_4\right]K_4 = 0,$$

$$\left[(Q_1^0 - S_{33}^0)m_3 + Q_1^0 s_3\right]K_3 + \left[(Q_1^0 - S_{33}^0)m_4 + Q_1^0 s_4\right]K_4 = 0, \quad (37)$$

$$\left[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3\right]K_3 + \left[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4\right]K_4 = 0$$

If we put $Q_2^0 = Q_1^0$ and $S_{11}^0 = S_{33}^0$ then first two equation of (37) are identical and hence (37) reduces to

$$\left[(Q_2^0 + P^0/2)s_3 + (Q_2^0 - R^0/2)m_3\right]K_3 + \left[(Q_2^0 + P^0/2)s_4 + (Q_2^0 - R^0/2)m_4\right]K_4 = 0,$$

$$\left[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3\right]K_3 + \left[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4\right]K_4 = 0 \quad (38)$$

The elimination of K_3 and K_4 from (38) gives

$$\begin{vmatrix} (Q_2^0 + P^0/2)s_3 + (Q_2^0 - R^0/2)m_3 & (Q_2^0 + P^0/2)s_4 + (Q_2^0 - R^0/2)m_4 \\ (B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3 & (B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4 \end{vmatrix} = 0, \quad (39)$$

which is the frequency equation of Rayleigh waves propagating over the free surface of heterogeneous prestressed medium.

CASE 1 – when the medium is heterogeneous unstressed (i.e. $P^0 = 0$) then the value of determinate (39) becomes

$$\begin{vmatrix} s_3 + m_3 & s_4 + m_4 \\ s_3 m_3 l - \lambda_0\sigma^2 & s_4 m_4 l - \lambda_0\sigma^2 \end{vmatrix} = 0 \quad (40)$$

CASE 2- when the medium is homogeneous isotropic elastic, the value of m from (29) and putting $a = 0$ in equation (28) gives the value of m_3 and m_4 as

$m_3 = \frac{-\sigma}{\sqrt{1-\frac{c^2}{\beta^2}}}$, $\quad m_4 = -\sigma\sqrt{1-\frac{c^2}{\alpha^2}}$ (41)

Using (23) and (41) in (40), we get

$$\left(2-\frac{c^2}{\beta^2}\right)^2 - 4\sqrt{1-\frac{c^2}{\alpha^2}}\sqrt{1-\frac{c^2}{\beta^2}} = 0 \quad (42)$$

giving the frequency equation of Rayleigh waves for the homogeneous case as it should.

Since σ and $s_6 \neq 0$, we must have $K_6 = 0$. The (31) reduce to

$u_1 = \sum_{i=3}^{4} K_i e^{s_i z}$, $v_1 = \frac{\eta}{\xi} u_1$,

$w_1 = \frac{i}{\xi}\sum_{i=3}^{4} m_i K_i e^{s_i z}$. (43)

On simplification, the determinately value of (39) becomes

$\frac{(s_3-s_4)\sigma^2}{D_3 D_4}\left[\frac{LD_3 D_4}{\sigma^2} - N\{(\lambda_0 l(s_3+s_4) + a^2\lambda_0 l + mE + m\, l s_3 s_4\}\right]=0$, (44)

where

$D_i = E - l s_i^2 - a\, l s_i$, (i =3, 4)

$L = A(Q_2^0 + P^0/2) + B_{22}^0(Q_2^0 - R^0/2) m_3 m_4$,

$N = A(Q_2^0 - R^0/2) + B_{22}^0(Q_2^0 + P^0/2) s_3 s_4$. (45)

$A = (B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2$.

We have now to interpret the frequency equation (44) subject to condition

$\mathrm{Re}(s_1, s_2) > a$

D_3, $D_4 \neq 0$ (46)

It is clear from equation that the factor $(s_3 - s_4) = 0$ of (44) gives only a trivial solution.

Put $s_3 = s_4 = s'$ (say) (47)

Substituting the values of s_4 and s_3 from (20) in $s_4 - s_3 = 0$, we get

$[(p_1-p_2)^2 - 4p_1]^{\frac{1}{2}} = 0$, (48)

where

$C = \frac{p}{\sigma} = \frac{2ap^2(1-2\gamma^2)^{\frac{1}{2}}}{p(1-\gamma^2)}$ (49)

For c to be real, $1-2\gamma^2 > 0$, i.e.

$\lambda_0 > 0$ (50)

In this case, from (43), we get

$u_1 = (K_3 + k_4) e^{s' z}$,

$v_1 = \frac{\eta}{\xi} u_1$, (51)

$w_1 = \frac{im'}{\xi} u_1$

Using in (38), we get

$s_3 = s_4 = s'$ and $m_3 = m_4 = m'$

$[(Q_2^0 + P^0/2)\, s' + (Q_2^0 - R^0/2)\, m'][K_3 + k_4] = 0$,

$[A - B_{22}^0\, m' s'][K_3 + k_4] = 0$. (52)

This equation can have a non zero solution for $K_3 + k_4$ only if

$(Q_2^0 + P^0/2)\, s' + (Q_2^0 - R^0/2)\, m' = 0$

and (53)

$A - B_{22}^0\, m' s' = 0$

Or

$A + B_{22}^0\, m' s' \left(\frac{Q_2^0 + P^0/2}{Q_2^0 - R^0/2}\right) = 0$ (54)

This, however, cannot hold in the present case, since from equation (50) $\lambda_0 > 0$ and so $B_{22}^0 > 0$ and s' is real. Hence the only solution of equation (52) is $K_3 + K_4 = 0$, giving the trival solution $u = v = w = 0$ from equation (51).

Here we consider, the values from (22)

$B_{22}^0 = \lambda_0 + 2\mu_0$, $B_{12}^0 = B_{23}^0 = \lambda_0$, (55)

$A_2^0 = Q_1^0 = Q_2^0 = \mu_0$ and putting $\lambda_0 = 0$,

The frequency equation to be studied from (44) becomes

$\frac{(E - l s_3^2 - a l s_3)(E - l s_4^2 - a l s_4)(S_{11}^0 \xi^2 + S_{33}^0\eta^2)(\mu_0 + \frac{P^0}{2})}{\sigma^2} + 2\mu_0\sigma^2 m^2 \left(\mu_0 - \frac{R^0}{2}\right) s_3 s_4 - m\{(S_{11}^0\xi^2 + S_{33}^0\eta^2)\left(\mu_0 - \frac{R^0}{2}\right) + 2\mu_0\left(\mu_0 + \frac{P^0}{2}\right)s_3 s_4\}\{E + l s_3 s_4\} = 0$ (56)

It is clear from (56), the frequency equation for Rayleigh waves depends on initial stress also.

Again we put $S_{11}^0 = S_{33}^0$ the frequency (56) becomes

$\{E - l s_3(s_3+a)\}\{E - l s_4(s_4+a)\} S_{11}^0 \mu_0 + 2\mu_0\sigma^2(\mu_0 - S_{11}^0) m^2 s_3 s_4 - m[\sigma^2 S_{11}^0(\mu_0 - S_{11}^0) + 2\mu_0^2 s_3 s_4][E + l s_3 s_4] = 0$

Putting $S_{11}^0 = S_{33}^0 = 0$, in above equation we get

$E + l s_3 s_4 - \sigma^2 m = 0$ (57)

In the rest of the analysis we will make use of following notations

$\frac{p^2}{\sigma^2} = C_R^2 = \frac{C_S^2}{\psi}$,

$\frac{p^2}{C_S^2} = \zeta$ so that $\sigma^2 = \zeta\,\psi$ and $\psi > 0$, $\zeta > 0$, (58)

C_R being the phase velocity of Rayleigh waves.

From (20) and (21), using (58), we obtain

$C_p^2 = 2C_S^2$, $p_3 = 0$ and $s_3 = \frac{-(s_1^* + a)}{2}$,

$s_4 = \frac{-(s_2^* + a)}{2}$ (59)

where $s_1^* = a^2 + 4\zeta(\psi - 1)$ and

$$s_2^* = a^2 + 4\zeta(\psi - 1) \qquad (60)$$

Using (59) in (57), we obtained

$$2s_3 s_4 = \zeta \text{ or } (s_1^* + a)(s_2^* + a) = 2\zeta \qquad (61)$$

From (59), a necessary condition (s_1^*, s_2^*) >a (here $s_1^* \text{ and } s_2^*$ are real) is that □>1.

From (60) and (61) after some calculation, we get

$$s_1^* = -\frac{a(4\psi^2 - 2\psi + 1)}{4\left(\psi - \frac{3-\sqrt{5}}{4}\right)\left(\psi - \frac{3+\sqrt{5}}{4}\right)}, \qquad (62)$$

changing (ψ -1) to (ψ - $\frac{1}{2}$) and vise versa in (62) we get

$$s_2^* = -\frac{a(4\psi^2 - 2\psi + 1)}{4\left(\psi - \frac{3-\sqrt{5}}{4}\right)\left(\psi - \frac{3+\sqrt{5}}{4}\right)}. \qquad (63)$$

Since ψ>1, (s_1^*, s_2^*)>a in the range 1< ψ< $\frac{3+\sqrt{5}}{4}$
(64)

Using equation (13), (20), (21), (58) in (45) and again putting $\lambda_0 = 0$ in these equations, on simplification, we get

$$D_3 = -\mu_0 \zeta(\psi - 1)$$

$$D_4 = -\mu_0 \zeta \psi \qquad (65)$$

From (60) and (62), we get

$$\omega = \frac{p}{2\Pi} = \frac{\sqrt{\zeta}\beta}{2\Pi} = \left(\frac{\sqrt{4a^2\psi(2\psi - 1)}}{(4\psi^2 - 6\psi + 1)}\right)\left(\frac{\beta}{2\Pi}\right) = \frac{a\beta\sqrt{\psi(2\psi - 1)}}{\Pi(4\psi^2 - 6\psi + 1)},$$
(66)

it is clear from (66) ω increase from $\frac{a\beta}{\Pi}$ $to\ \infty$ as ψ increases from 1 to $\frac{3+\sqrt{5}}{4}$. The value of cut off frequency below which Rayleigh waves cannot exists is $\frac{a\beta}{\Pi}$. At ψ=1 $C_R = \beta$ and at ψ = $\frac{3+\sqrt{5}}{4}$, $C_R = C_{R_0}$ where $C_{R_0}^2 = (3 - \sqrt{5})\beta^2 gives$ Rayleigh waves velocity of the homogeneous medium of the same Poisson's ratio.

V. For Numerical Calculations

To derive Rayleigh waves velocity 'c_0' of the homogeneous medium. Put $a = 0$ in (6), then Lame's constants and density correspond to the homogeneous medium. The condition (46) Re $(s_1^*, s_2^*) > a$ becomes Re ($s_1^*, s_2^*) > 0$, which is possible only for $a = 0$ iff $4\psi^2 - 6\psi + 1 = 0$ ($s_1^* \text{ and } s_2^*$ become indeterminate in (62),(63)), from here we found the value of $\psi = \frac{(3+\sqrt{5})}{4}$. We have drawn the graph between $\frac{c}{\beta}$ and $\frac{\pi\omega}{a\beta}$ on the variation of λ from 1 to $\frac{(3+\sqrt{5})}{4}$ as shown in figure 2.

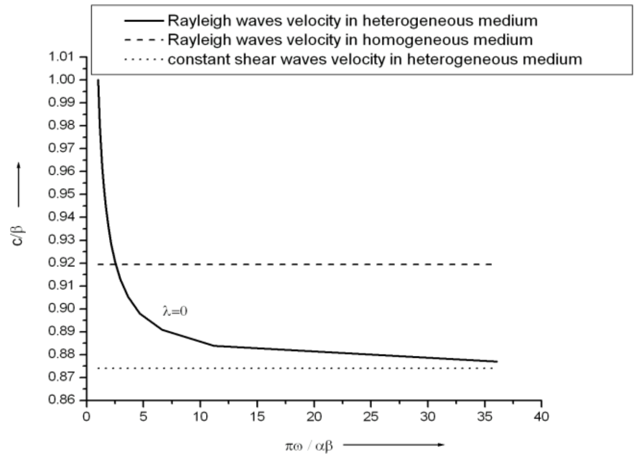

Figure 2

VI. Conclusion

The analysis of the frequency equation carried out for the case $\lambda = 0$, shows that, there exist single Rayleigh modes which cannot propagate below certain cut-off frequency .It is found that speed of Rayleigh waves lies in a limit $\beta < c < c_0$, where c, β denote the phase and constant shear waves velocity in heterogeneous medium and $c_0$is the corresponding Rayleigh waves velocity in homogeneous medium of the same Poisson's ratio.

It is clear from frequency equation of Rayleigh waves in heterogeneous prestressed medium depends not only initial stress but also inhomogeneity factor (a>0) .Here we discussed three cases given below.

Case 1. When the medium is prestressed homogeneous the frequency equation coincides with the results of Singh and Singh (1991).

Case 2. When the medium is heterogeneous unstressed ($P^0 = 0$) the frequency equation for Rayleigh waves coincide with the results of Sidhu (1970).

Case 3. When the medium is homogeneous unstressed then the frequency equation coincide with the classical results.

References

[1] A.Ben-Menahem and S.J. Singh, "Seismic waves and sources", Springer-verlag, New York, 1981.

[2] A.M. Abd-Alla, S.R. Mahmoud and M.I.R. Helmi,"Effect of initial stress and magnetic field on Propagation of shear wave in non-homogeneous anisotropic medium under gravity field", Open Appl. Math. J (3), 2009, pp. 49-56.

[3] A.N. Norries, "Propagation of plane waves in prestressed elastic medium", *J.*Acoust. Soc. Am., 1983, pp. 1642-1643.

[4] I. Singh and S. Singh, "Surface waves in prestressed elastic media", Acta. Geophys. Pol. 39 (1), pp. 133-145.

[5] J.F. Hook, "Separation of the vector Wave equation of elasticity for certain types of Inhomogeneous isotropic media", *J.*Acoust. Soc. Am.33 (3), 1961, pp. 302-313.

[6] J.T. Wilson, "Surface waves in a heterogeneous media", Bull. Seism. Soc. Am. **32**, 1942, pp. 297-304.

[7] L.M. Brekhovskikh, "Waves in layered media", Academic press, *New* York, 1960.

[8] K. Singh, "Propagation of Rayleigh waves in an axially symmetric heterogeneous layer lying between two homogeneous half space", pure and appl. Geophysics. **61**(1), 1965, pp. 36-44.

[9] Lord Rayleigh, "On waves propagated alone the plane surface of an elastic solid", Proc.Lond. Math. Soc. **17**, 1885, pp. 4-11.

[10] M.A. Biot, "Mechanics of incremental deformation", john Wiley *and* sons, New *York*, 1965.

[11] M.Newlands, "Rayleigh waves in a two layer heterogeneous medium", Monthly notice of Roy.Astron.Soc. Geophysical Suppl. 6(2), 1950, pp. 109-124.

[12] R. Kakar and S. Kakar, "Rayleigh waves in a non-homogeneous granular media", Chem. Bio. Phy. Sci. Sec. *B* **3**, 2013, pp. 464-478.

[13] R.S. Sidhu, "Transmission of Rayleigh waves over the surface of a heterogeneous medium", Pure and appl. Geophysics**,** 1, 1970, pp. 48-70.

[14] R.S. Sidhu and S.J. Singh, "Comments on reflection elastic waves under initial stress at a free Surface: P and SV motion [J.Acoust.Soc.Am.72 (1982) 255-263(1982)]", *J.*Acoust.Soc.Am.74**,** 1983 , pp. 1940-1942.

[15] S. Dutta, "Rayleigh waves in two layer heterogeneous medium", Bull. Seism. Soc. Am. 53(5), 1963, pp. 1007-1022.

[16] S.C. Das, D.P. Acharya and P.R. Sengupta, "Surface waves in an inhomogeneous Elastic medium under the influence of gravity", Rev.Roum. Des. Sci. Tech. 37**,** 1992, pp. 539–551.

[17] R. Stoneley, "The transmission of Rayleigh waves in a heterogeneous medium", monthly notices of the Roy Astron. Soc. Geophysical Suppl. 3(6), 1934, pp. 222-232.

[18] T. Karlsonand and J.F. Hook, "Lamb's problem for an inhomogeneous medium with constant velocities of propagation", Bull. Seism. Soc. Am.53 (5), 1963, pp. 1007-1022.

[19] W.L. Pilant, "Elastic Waves in the Earth", Elsevier sci. publ. Comp. Amsterdam, 1979R. J. Vidmar. (1992, August). On the use of atmospheric plasmas as electromagnetic reflectors. *IEEE Trans. Plasma Sci.* [Online]. *21(3).* pp. 876—880. Available: http://www.halcyon.com/pub/journals/21ps03-vidmar