Global Existence of Classical Solutions for A Class Nonlinear Parabolic Equations

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***Abstract*— In this article we prove the existence of classical solutions for a class nonlinear parabolic equations. We propose new integral representation of the classical solutions. As an application we give continuous dependence and differentiability of the solutions with respect to the initial data and parameters.**

***Index Terms*— parabolic equation, existence, dependence on initial data,**

I. Introduction

In this article we investigate the Cauchy problem

$u_t \; u_{xx} = f(t; x; u; u_x)$ in $(0; \infty) * R$, (1.1)

$u(0; x) = \Phi(x)$ in R; (1.2)

Where Φ € $C^2(R)$, $f : [0; \infty) *R* R* R \longrightarrow R$ is a given continuous function, $u : [0; \infty)*R \longrightarrow R$ is unknown.

Our main result is as follows.

Theorem 1.1. Let f € $C[0; \infty) *R* R* R)$, Φ € $C^2(R)$. Then the problem (1.1), (1.2) has a solution u € $C^1[0; \infty); C^2(R))$.

To prove our main result we propose a new approach different than the well-known approaches. Also, we propose new integral representation of the solutions of the initial value problem (1.1), (1.2).

As an application of our new integral representation we deduct some results connected with the continuous dependence on the initial data and parameters of the problem (1.1), (1.2).

Theorem 1.2. Let f € $C[0; \infty) *R* R* R)$, $\frac{\delta f}{\delta u}, \frac{\delta f}{\delta u_x}$ exist and are continuous in $[0; \infty)*R* R* R$, Φ € $C^2(R)$. Let also $u(t,x,0, \Phi)$ € $C^1[0; \infty); C^2(R))$ be a solution to the problem (1.1), (1.2). Then $u(t,x,0, \Phi)$ is differentiable with respect to Φ and $u(t,x) = \frac{\delta u}{\delta \phi}(t,x,0,\Phi)$ satisfies the following initial value problem

$$u_t - u_{xx} = \frac{\delta f}{\delta u}\left(t,x,u(t,x,0,\Phi),u_x(t,x,0,\Phi)\right)u + \frac{\delta f}{\delta u_x}\left(t,x,u(t,x,0,\Phi),u_x(t,x,0,\Phi)\right)u_x \text{ in} [0;\infty) *R , \quad (1.3)$$

$u(0,x) = 1$ in R (1.4)

II. Auxiliary results

We will start with the following important lemma.

Lemma 2.1. Let f € $C([a, b] *[c, d]* R* R)$, g € $C^2([c; d])$. Then the function u € $C^1([a, b]; C^2([c,d]))$ is a solution to the problem

$u_t - u_{xx} = f(t, x, u, u_x)$ in $(a,b]*[c,d]$, (2.1)

$u(a,x) = g(x)$ in $[c,d]$ (2.2)

if and only if it is a solution to the integral equation

$$\int_c^x \int_c^y \big(u(t,z) - g(z)\big)dzdy - \int_a^t (u(\tau,x) - u(\tau,c) - (x - c)ux(\tau,c))d\tau$$
$$= \int_a^t \int_c^x \int_c^y f\big(\tau,z,u(\tau,z),u_x(\tau,z)\big)dzdyd\tau, \quad x€ [c,d],t €[a,b] \quad (2.3)$$

A. Proof. 1.

Let u € C^1 ([a,b],C^2([c,d])) is a solution to the problem (2.1),(2.2).

We integrate the equation (2.1) with respect to x and we get

$$\int_c^x u_t(t,z)dz - \int_c^x u_{xx}(t,z)dz = \int_c^x f\big(t,z,u(t,z),u_x(t,z)\big)dz, \quad x € [c,d],t €[a,b]$$

$$\int_c^x u_t(t,z)dz - u_x(t,z) + u_x(t,c) = \int_c^x f\big(t,z,u(t,z),u_x(t,z)\big)dz, \quad x € [c,d],t €[a,b]$$

Now we integrate the last equation with respect to x and we find

$$\int_c^x \int_c^y u_t(t,z)dzdy - \int_c^x u_x(t,z) - u_x(t,c))dz = \int_c^x \int_c^y f\big(t,z,u(t,z),u_x(t,z)\big)dzdy, \quad x € [c,d],t €[a,b]$$

or

$$\int_c^x \int_c^y u_t(t,z)dzdy - u(t,x) + u(t,c) + (x-c)u_x(t,c) = \int_c^x \int_c^y f\big(t,z,u(t,z),u_x(t,z)\big)dzdy, \quad x € [c,d],t €[a,b]$$

Now we integrate the last equality with respect to t and we obtain

$$\int_a^t \int_c^x \int_c^y u_t(s,z)dzdyds - \int_a^t \big(u(s,x) - u(s,c) - x-cuxs,cds = \int_a^t \int_c^x \int_c^y f\big(s,z,u(s,z),u_x(s,z)\big)dzdyds, \quad x € [c,d],t €[a,b]$$

or

$$\int_c^x \int_c^y u(t,z) - g(z))dzdy - \int_a^t u(s,x) - u(s,c) - x-cux(s,c))ds = \int_a^t \int_c^x \int_c^y f\big(s,z,u(s,z),u_x(s,z)\big)dzdyds, \quad x € [c,d],t €[a,b]$$

i.e. , u satisfies the equation(2.3).

B. Proof 2

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Let u €C^1([a, b], C^2([c, d])) be a solution to the integral equation (2.3). We differentiate the equation (2.3) with respect to x and we get

$$\int_c^x \int_c^y u(t,z) - g(z))dz - \int_a^t (u_x(s,x) - u_x(s,c))ds$$
$$= \int_a^t \int_c^x \int_c^y f\big(s,z,u(s,z),u_x(s,z)\big)dzdyds,$$
x € [c,d],t €[a,b]

Again we differentiate with respect to x and we find

$$u(t,x)-g(x) - \int_a^t u_{xx}(s,x)ds$$
$$= \int_a^t f\big(s,x,u(s,x),u_x(s,x)\big)ds, \qquad x € [c,d],t €[a,b]$$

Now we put t=a in the last equation and we find

U(a,x) = g(x), x€[c,d],

i.e., the function u satisfies (2.2).

Now we differentiate the equation (2.4) with respect to t and we find

$$u_t(t,x) - u_{xx}(t,x) = f\big(t,x,u(t,x),u_x(t,x)\big),$$
x € [c,d],t €[a,b]

The proof of the existence result is based on the following theorem.

Theorem 2.2. [1] Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that

1. S is continuous and S(x) resides in a compact subset of Y .

2. T : X ➞Y is expansive and onto.

Then there exists a point x^* € X such that

$$Sx^* + T x^* = x^*$$

Definition 2.3. Let (X; d) be a metric space and M be a subset of X. The mapping T : M ➞X is said to be expansive if there exists a constant h > 1 such that

$$d(T_x, T_y) \geq hd(x; y)$$

for any x, y €M.

III. PROOF OF THE EXISTENCE RESULT

Step 1. Firstly, we will prove that the problem

$$u_t - u_{xx} = f(t,x, u, u_x) \quad \text{in} \quad (0, 1] [0, 1], \qquad (3.1)$$

$$u(0, x) = \Phi(x) \quad \text{in} \quad [0, 1] \qquad (3.2)$$

has a solution u € C^1([0; 1]; C2([0; 1])).

Let E11 = C1([0,1], C^2([0,1])) be endowed with the norm

$$||u|| = \max\{\max_{t,x€[0,1]}|u(t,x)|, \quad \max_{t,x€[0,1]}|u_t(t,x)|$$
$$\max_{t,x€[0,1]}|u_x(t,x)|, \quad \max_{t,x€[0,1]}|u_{xx}(t,x)|\}$$

With K_{11} we donate the set of all equicontinuous families in E_{11}, i.e., for every ε>0 there exist δ(ε) > 0 such that

$|u(t_1,x_1) - u(t_2,x_2)| < ε$, $|u_t(t_1,x_1) - u_t(t_2,x_2)| < ε$,

$|u_x(t_1,x_1) - u_x(t_2,x_2)| < ε$, $|u_{xx}(t_1,x_1) - ux_x(t_2,x_2)| < ε$

Whenever $| t_1-t_2| < δ$, $| x_1-x_2| < δ$.

Let B > 0 be arbitrarily chosen, $K'_{11} = \overline{K_{11}}$,

$K_{11} = \{u€, K'_{11} : ||u|| \leq B\}$.

Since Φ € C([0,1]), f € C([0, 1]* [0, 1]* [- B, B] *[- B,B]) we have that there exists a constant $M_{11} > 0$ such that

$|\Phi(x)| \leq M_{11}$ in [0,1],

$|f(t,x,y,z)| \leq M_{11}$ in [0, 1]* [0, 1]* [- B, B] *[- B,B]

Let l>0 be chosen so that

$$L(5B + 2 M_{11}) \leq B \qquad (3.3)$$

Let also,

$$L_{11} = \{u € K'_{11} : ||u|| \leq (1+l) B\}.$$

We note that K_{11} is a closed convex subset of L_{11}

For u € L_{11} we define the operators

T_{11} (u) (t,x) = (1+l) u(t,x),

S_{11} (u) (t,x) = -lu(t,x) + l $\int_0^x \int_0^y \big(u(t,z) - \Phi(z)\big)dzdy -$ *l0tuτ,x−uτ,o−xuxτ,0dτ−l0t0x0yf(τ,z,uτ,z,ux(τ,z))dzdy dτ.*

a) $S_{11} : K_{11}$ ➞ K_{11}. Let u € K_{11}. Then $S_{11}(u)$ € C^1 ([0,1], C^2([0, 1])) and for (t, x) 2 [0, 1] [0, 1], using the choice (3.6) of the constant l, we have

$|S_{11}$ (u) (t,x)$|$ = $\Big|-lu(t,x) + l \int_0^x \int_0^y \big(u(t,z) -$ *Φzdzdy l0tuτ,x−uτ,0−xuxτ,0dτ−l 0t0x0yfτ,z,uτ,zuxτ,zdzdydτ|,*

$\leq l|u(\tau,x)| + l\int_0^x \int_0^y (|u(t,z)| + |\Phi(z)|)dzdy +$ *l0t(uτ,x+uτ,0+x|uxτ,0|)dτ+l0t0x0y|f(τ,z,uτ,z,uxτ,z)|dzdydτ ≤lB+lB+M11+3lB+l M11=5B+2M11l≤B,*

$S_{11}(u)_t(t,x) = -lu_t(t,x) + l\int_0^x \int_0^y u_t(t,z)dzdy -$ *lut,x−ut,0−xuxt,0−l0x0yft,z,ut,z,uxt,zdzdy*

$\leq l|u_t(t,x)| + l\int_0^x \int_0^y |u_t(t,z)|dzdy + l(|u(t,x)| +$ *ut,0+xuxt,0+l0x0yft,z,ut,z,uxt,zdzdy*

$\leq lB + lB + 3lB + lM_{11} = l(5B + M_{11}) \leq B$,

$S_{11}(u)_x(t,x) = -lu_x(t,x) + l\int_0^x \big(u(t,z) - \Phi(z)\big)dz -$ *l0t(uxτ,x− uxτ,0)dτ−l0t0xf(τ,z,uτ,z,uxτ,z))dzdτ,*

$|S_{11}(u)_x(t,x)| = | - lu_x(t,x) + l\int_0^x \big(u(t,z) - \Phi(z)\big)dz -$ *l0t(uxτ,x− uxτ,0)dτ−l0t0xf(τ,z,uτ,z,uxτ,z)dzdτ|,*

$\leq l|u_t(t,x)| + l\int_0^x (|u(t,z)| + \Phi(z)|dz +$ *l 0t(|ux(τ,x)|+uxτ,0dτ+l0t0x|f(τ,z.uτ,z, uxτ,z)|dzdτ*

$\leq lB + l(B + M_{11}) + 2lB + lM_{11} = l(4B + 2M_{11}) \leq B$,

$S_{11}(u)_{xx}(t,x) = -lu_{xx}(t,x) + l\big(u(t,x) - \Phi(x)\big) -$
$l\int_0^t (u_{xx}(\tau,x)d\tau - l\int_0^t f\big(\tau,x,u(\tau,x),u_x(\tau,x)\big)d\tau,$

$|S_{11}(u)_{xx}(t,x)| = | - lu_{xx}(t,x) + l\big(u(t,x) - \Phi(x)\big) +$
$l\int_0^t (u_{xx}(\tau,x)d\tau - l\int_0^t f\big(\tau,x,u(\tau,x),u_x(\tau,x)\big)d\tau|,$

$\leq lB + l(B + M_{11}) + lB + lM_{11} = l(3B + 2M_{11}) \leq B$,

We note that $\{S_{11}(u) : u € K_{11}\}$ is an equicontinuous family in E_{11}. Consequently $S_{11} : K_{11}$➞ $K_{11}\}$. Also, $S_{11}(K_{11}) \subset K_{11} \subset L_{11}$, i.e., $S_{11}(K_{11})$ resides in a compact subset of L_{11}.

b) S_{11}: K_{11} → K_{11} is a continuous operator. We note that if $\{u_n\}\infty_{n=1}$ be a sequence of elements of K_{11} such that u_n → u in K_{11} as n → ∞, then $S_{11}(u_n)$→ $S_{11}(u)$ in K_{11} as n → ∞
Therefore S_{11} : K_{11} → K_{11} is a continuous operator.
c) T_{11} : K_{11} →L_{11} is an expansive operator and onto. For u,v € K_{11} we have that

$$||T_{11}(u) - T_{11}(v)|| = (1+l)\ ||u-v||,$$

i.e., T_{11} : K_{11} →L_{11} is an expansive operator with constant 1+l.
Let v € L_{11}. Then v\1+l € K_{11} and

$$T_{11}(v\backslash 1+l) = v,$$

i.e. , T_{11} : K_{11}→ L_{11} is onto.
From a), b), c) and from theorm 2.2, it follows that there is u_{11}€ K_{11} such that

$$T_{11}\, u_{11}+ S_{11}\, u_{11} = u_{11}$$

Or
$(1 + l)u_{11}(t,x) - lu_{11}(t,x) + l\int_0^x\int_0^y(u_{11}(t,z) -$
$\Phi(z))dzdy - l\int_0^t u_{11}(\tau,x) - u_{11}(\tau,0) - xu_{11x}(\tau,0))d\tau -$
l 0t0x0yf(τ,z,u11(τ,z),u11x(τ,z))dzdydτ=u11(t,x),
Or
$\int_0^x\int_0^y(u_{11}(t,z) - \Phi(z))dzdy - l\int_0^t u_{11}(\tau,x) - u_{11}(\tau,0) -$
xu11xτ,0)dτ−l
0t0x0yf(τ,z,u11(τ,z),u11x(τ,z))dzdydτ=0,

Whereupon, using Lemma 2.1, we conclude that
u_{11}€ $C^1([0,1], C^2([0,1]))$ is a solution to the problem (3.1), (3.2).

Step 2. Now we consider the problem

$$u_t-u_{xx} = f(t,x,u(,x),u_x(t,x)) \quad \text{in} \quad (0,1]*[1,2], \qquad (3.4)$$
$$u(0,x) = \Phi(x) \quad \text{in} \quad [1,2] \qquad (3.5)$$

Let $E_{12} = C^1([0,1], C^2([1,2]))$ be endowed with the norm

$$||u|| = \max\{\max_{t,x\in[0,1]*[1,2]}|u(t,x)|,\quad \max_{t,x\in[0,1]*[1,2]}|u_t(t,x)|$$
$$\max_{t,x\in[0,1]*[1,2]}|u_x(t,x)|,\quad \max_{t,x\in[0,1]*[1,2]}|u_{xx}(t,x)|\}$$

With K_{12} we denote the set of all equicontinuous families in E_{12}. Let $K'_{12} = \overline{K_{12}}$,

$$K_{12} = \{u\text{€ } K'_{12}\ ||u|| \le B\}$$

Since Φ € C([1,2]), f € C([0,1]*[1,2]*[-B,B]*[-B,B]) we have that there exists a constant M_{12} >0 such that

$$|\Phi(x)| \le M_{12} \quad \text{in} \quad [1,2],$$
$$|f(t,x,y,z)| \le M_{12}$$

in [0,1] X [1,2] X [-B,B] X [-B, B].
Let l_1 >0 be chosen so that

$$l_1\,(5B+2M_{12}) \le B. \qquad (3.6)$$

Let also,

$$L_{12} = \{u \in K' : ||u|| \le (1 + l_1)B\}$$

We note that K_{12} is a closed convex subset of L_{12}.
For u Є L_{12} we define the operators
$T_{12}(u)(t,x)$= $(1 + l_1)u(t,x)$,

$$S_{12}(u)(t,x)$$
$$= -l_1u(t,x) + l_1\int_1^x\int_1^y(u(t,z) - \emptyset(z))dzdy$$
$$- l_1\int_0^t(u(\tau,x) - u_{11}(\tau,1) - (x-1)u_{11x}(\tau,1))d\tau$$
$$- l_1\int_0^t\int_1^x\int_1^y f(\tau,z,u(\tau,z),v_x(\tau,z))dzdyd\tau$$

As in Step 2one can prove that there is u_{12} Є $C^1([0,1],C^2([1,2]))$ which is a solution to the problem(3.4), (3.5). This solution u_{12} satisfies the integral equation

$$\int_1^x\int_1^y(u_{12}(t,z) - \emptyset(z))dzdy \qquad (3.7)$$
$$- \int_0^t(u_{12}(\tau,x) - u_{11}(\tau,1) - (x-1)u_{11x}(\tau,1))d\tau$$
$$- \int_0^t\int_1^x\int_1^y f(\tau,z,u_{12}(\tau,z),u_{12x}(\tau,z))dzdyd\tau = 0$$

Now we put x=1 in (3.7) and we find

$$\int_0^t(u_{12}(\tau,1) - u_{11}(\tau,1)d\tau = 0,$$

Which we differentiate with respect to t and we get

$$u_{12}(t,1) = u_{11}(t,1)$$

in [0,1]. (3.8)
Now we differentiate (3.7) with respect to x and we find
$\int_x^1(u_{12}(t,z) - \Phi(z))dz - \int_0^t(u_{12x}(\tau,x) - u_{11x}(\tau,1))d\tau -$
$\int_0^t\int_1^x f(\tau,z,u_{12}(\tau,z),u_{12x}(\tau,z))dzd\tau = 0$
In the last equation we put x=1 and we become
$\int_0^t(u_{12x}(\tau,x) - u_{11x}(\tau,1))d\tau = 0,$
Which we differentiate with respect to t and we get

$$u_{12x}(t,1) = u_{11x}(t,1) \quad \text{in} \quad [0,1] \qquad (3.9)$$

Now we differentiate (3.8) with respect to t and we get

$$u_{12t}(t,1) = u_{11t}(t,1) \quad \text{in} \quad [0,1].$$

Hence , (3.8),(3.9) and
f(t,1, $u_{11}(t,1), u_{11x}(t,1)) = f(t,1,u_{12}(t,1),u_{12x}(t,1))$,
we find

$$u_{12xx}(t,1) = u_{12t}(t,1) - f(t,1,u_{12}(t,1),u_{12x}(t,1))$$
$$= u_{11}(t,1) - f(t,1,u_{11}(t,1),u_{11x}(t,1)) = u_{11xx}(t,1)$$

in [0,1].
Consequently the function

$$U(x,t) = \begin{cases} u_{11}(t,x) & in \quad [0,1]*[0,1] \\ u_{12}(t,x) & in \quad [0,1]*[1,2], \end{cases}$$

$C^1([0,1], C^2([0,2]))$ is a solution to the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \quad \text{in} \quad (0,1]*[0,2],$$
$$u(0,x) = \Phi(x) \quad \text{in} \quad [0,2].$$

Then we consider the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \quad \text{in} \quad (0,1]*[2,3] \qquad (3.10)$$
$$u(0,x) = \Phi(x) \quad \text{in} \quad [2,3].$$

As in above there is u_{13} € $C^1([0,1], C^2([2,3]))$ which is a solution to the problem (3.10) and satisfies the integral equation

$\int_2^x\int_2^y(u_{13}(t,z) - \Phi(z))dzdy - \int_0^t(u_{13}(\tau,x) -$
(u12τ,2−x−2u12x(τ,2))dτ−0t2x2yfτ,z,u13τ,z,u13xτ,z dzdydτ=0

The function

$$\begin{cases} u_{11}(t,x) & in \quad [0,1]*[0,1] \\ u_{12}(t,x) & in \quad [0,1]*[1,2] \\ u_{13}(t,x) & in \quad [0,1]*[2,3] \end{cases}$$

$C^1([0,1], C^2([0,3]))$ is a solution to the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \quad \text{in} \quad (0,1]*[0,3],$$
$$u(0,x) = \Phi(x) \quad \text{in} \quad [0,3].$$

An so on. We construct a solution u_{13} € $C^1([0,1], C^2R))$ which is a solution to the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \quad \text{in} \quad (0,1]*R,$$
$$u(0,x) = \Phi(x) \quad \text{in} \quad R.$$

Then we consider the problem

$$u_t - u_{xx} = f(t, x, u(t,x), u_x(t,x)) \quad \text{in} \quad (1,2]*[0,1],$$
$$u(1,x) = u_1(t,x) \quad \text{in} \quad [0,1].$$

As in above, this problem has a solution $u_{21} \in C^1([1,2], C^2([0,1]))$ which satisfies the integral equation

$\int_0^x \int_0^y \big(u_{21}(t,z) - u_1(1,z)\big)dzdy -$
1t(u21τ,x−u1τ,0−xu1τ,0)dτ−1t0x0yfτ,z,u21τ,z,u21xτ,zdzdydτ= 0.

We have that

$u_{21}(t,0) = u_1(t,0), \qquad u_{21t}(t,0) = u_{1t}(t,0),$

$u_{21x}(t,0) = u_{1x}(t,0), \quad u_{21xx}(t,0) = u_{1xx}(t,0)$ in [1,2]

Also, we have, $x \in [0,1]$,

$u_{21}(1,x) = u_1(1,x) = u_{11}(1,x), \quad u_{21t}(1,x) = u_{1t}(1,x) = u_{11t}(1,x),$

$u_{21x}(1,x) = u_{1x}(1,x) = u_{11x}(1,x), \quad u_{21xx}(1,x) = u_{1xx}(1,x) = u_{11xx}(1,x).$

An so on, we construct a solution $u \in C^1([0,2], C^2R))$ of the problem

$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x))$ in $(0,2]*R$,

$u(0,x) = \Phi(x)$ in R.

And so on, we construct a solution $u \in C^1([0,\infty), C^{2(}R))$ to the problem (1.1), (1.2).

We note that u_{mm} satisfies the integral equation

$$\int_{n-1}^{x}\int_{n-1}^{y}\big(u_{mn}(t,z) - u_{m-1}(m-1,z)\big)dzdy - \int_{m-1}^{t}\Big(u_{mn}(\tau,x) - u_{m-1}(\tau,n-1) - \big(x-(n-1)\big)u_{m-1}(\tau,n-1)\Big)d\tau - \int_{m-1}^{t}\int_{n-1}^{x}\int_{n-1}^{y} f(\tau,z,u_{mn}(\tau,z), u_{mnx}(\tau,z))dzdyd\tau = 0.$$

IV. Proof of Theorem 1.2

We have that the solution $u(t,x,0,\Phi)$ satisfies the following integral equation

$Q(\Phi) = \int_0^x \int_0^y \Big(u\big(t,z,0,\Phi(z)\big) - \Phi(z)\Big)dzdy -$
0t(u(τ,x,0,Φ(x))− u(τ,0,0,Φ(0))−xux(τ,0,0,Φ(0))dτ−
$\int_0^t \int_2^x \int_2^y f\big(\tau,z,u\ \ (\tau,z,0,\Phi(z)),u_x(\tau,z,0,\Phi(z))\big)dz = 0$

Then

$Q(\Phi) - Q(\Phi_1) =$
$\int_0^x \int_0^y \Big(u\big(t,z,0,\Phi(z)\big) - u\big(t,z,0,\Phi(z)\big)\Big) - \big(\Phi(z) -$
Φ1zdzdy0tuτ,x,0,Φx−uτ,x,0,Φ1xdτ+0tuτ,0,0,Φ0−uτ,0,0,Φ10dτ+xuxτ,0,0,Φ0−uxτ,0,0,Φ10)dτ−0t0x0yfτ,z,uτ,z,0,Φz,uxτ,z,0,Φ1zdz−

$f\big(\tau,z,u(\tau,z,0,\Phi_1(z)),u_x(\tau,z,0,\Phi_1(z))\big)dzdyd\tau$
$=$
$\int_0^x \int_0^y \Big(\frac{\partial u}{\partial\Phi}\big(t,z,0,\Phi(z)\big) - 1\Big)dzdy - \int_0^t \frac{\partial u}{\partial\Phi}(\tau,x,0,\Phi)d\tau +$
0t∂u∂Φτ,0,0,Φdτ+ 0tx∂u∂Φxτ,0,0,Φdτ−
$\int_0^t \int_0^x \int_0^y \frac{\partial f}{\partial u}\big(\tau,z,u(\tau,z,0,\Phi),u_x(\tau,z,0,\Phi)\big)\frac{\partial u}{\partial\Phi}(\tau,z,0,\Phi)$

$dzdyd\tau -$
$\int_0^t \int_0^x \int_0^y \frac{\partial f}{\partial u_x}\big(\tau,z,u(\tau,z,0,\Phi),u_x(\tau,z,0,\Phi)\big)\Big(\frac{\partial u}{\partial\Phi}\Big)_x(\tau,z,0,\Phi)dzdyd\tau +$
$\partial\{\Phi,\Phi_1\},$

Where $\partial\{\Phi,\Phi_1\} \longrightarrow 0$ as $\Phi(x) \longrightarrow \Phi_1(x)$ for every $x \in R$. Hence, when $\Phi(x) \longrightarrow \Phi_1(x)$ for every $x \in R$, we get

$0 =$
$\int_0^x \int_0^y (v(t,z) - 1)dzdy - \int_0^t v(\tau,x)d\tau + \int_0^t v(\tau,0)d\tau +$
0txvxτ,0dτ−
0t0x0y∂f∂uτ,z,uτ,z,0,Φ,uxτ,z,0,Φvτ,zdzdydτ−
0t0x0y∂f∂uτ,z,uτ,z,0,Φ,uxτ,z,0,Φvxτ,zdzdydτ (4.1)

Which we differentiate twice in x and once in x and we get that v satisfies (1.3). Now we put t=0 in (4.1) and then we differentiate twice in x, and we find that u satisfies (1.4).

References

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