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Asymptotic Behavior of a Class of Stochastic Differential Equations

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Abstract-- **In this article, we address some conditions on invariant measure of Markov semigroups which ensures stochastic bifurcations of a wide class of stochastic differential equations with fractional Brownian motion. This leads to sufficient conditions on Hurst parameter,drift and diffusion coefficients for a stochastic bifurcations of the families of random dynamical systems.According to the Hölder coefficient of the diffusion function around the singular point and Hurst parameter, we identify different regimes. Interestingly, for the first time it is found that the Hurst parameter affects both bifurcation conclusions and large deviations which is significantly different from the classical Brownian motion process. This fact is due to the long-range dependence (LRD) property of the fBm.**

Index Terms: **Random dynamical systems; Stochastic stability; Stochastic bifurcation; Invariant measure; Fractional Brownian motion**

I. Introduction

The classical central limit theorem (CLT) reveals that the probability distribution of the sum (or average) of many independent and identically distributed (i.i.d.) random variables with finite variance approaches the normal distribution[1-3].For this reason Gaussian models have been widely employed in many fields, and properties of Gaussian processes are characterized by second-order statistics, such as variance and correlation.In practical applications, however, since in many problems related to network traffic analysis, mathematical finance, and many other fields the processes under study seem empirically to exhibit the selfsimilar properties, and the long-range dependent properties, and since the fractional Brownian motions are the simplest processes of this kind[4-10]. It is suggested that [11] the fractional processes provide a better description of these random variables with properties of being infinite variance, non-Gaussian or non-stationary. Moreover, it is worth noting that the noise term in the stochastic system is used to describe the interaction between the (small) system and its (large) environment. The non-independence over disjoint time intervals in noise term applied by the environment to the system makes the fBm more useful since it can exhibit long-range dependence. For this reason, and also because the fBm includes the fundamental classical Brownian motion as a special case when $H=1/2$, there has been an increasing interest in the research activity related to the fBm itself and the stochastic systems driven by it.

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A large class of phenomena can be described using stochastic differential equations(SDE) drive white noise, in such distinct domains as economics, physics or biology. Many analytical methods have been developed to characterize the solutions of such systems. In particular, the widely applied theory of stochastic bifurcations[12] allows a qualitative characterization of the asymptotic regimes of random dynamical systems. This theory is very handy to study nonlinear stochastic differential equations and is used to characterize the asymptotic behavior of complicated systems. Investigating the impact of fractional Brownian motion noisy perturbations on such systems is hence of great interest, and is currently an active field of research. In recent years, it has been shown in many different areas that applying fractional noise on a random system can lead to many counter-intuitive phenomena, such as fractional noise-induced stabilization [13] and control[14]. From a mathematical perspective, understanding the interplays between white noise and fractional noisy perturbations is a great challenge, with many applications. Several tools have been introduced, ranging from the theory of random dynamical systems (RDS) [15,16], the study of moment equations [17,18] or multiscale stochastic methods for slow-fast systems [19]. Unfortunately no systematic method in the flavor of bifurcation theory for the analysis of the dynamics of nonlinear SDE drive fractional white noise exist, and this is the central problematic of the present manuscript.

We focus here on the dynamics of random dynamical systems induced parametrized one dimensional stochastic differential equation driven by the fractional Brownian motion(fBm). The question we address is how the interplay between the Hurst parameter, drift and the shape of the diffusion function affects the behavior of the system. An important contribution of RDS theory to the field of stochastic bifurcations is to distinguish between phenomenological(P) bifurcations and dynamical (D) bifurcations. One is the phenomenological approach favored by physicists and engineers based on the qualitative changes of stationary measure,i.e., the stationary probability density of the response. The other is the dynamical approach favored by mathematicians based on the qualitative changes of stability of invariant measures and occurrence of new invariant measures for random dynamical systems. Each approach has its advantages and the two approaches can be regarded as complementary to each other [12,20]. We investigate the questions of stochastic stability and bifurcations(P and D) in a comprehensive manner, using combinations of appropriate tools such as invariant measure, Lyapunov exponent, stationary measure and Lyapunov functionals. Mathematically, the choice of a vanishing diffusion coefficient provides a framework to study subtle competitions between the drift and the fractional noise, leading to a rich and generic phenomenology.

The paper is organized of as follows: In Section 2, we give some Definitions. We study the stochastic bifurcation in Section 3 and the two examples are given to illustrative our presented results in Section 4.

III. PRELIMINARIES

In the section, we present some preliminary results to be used in a subsequent section to establish the stochastic stability and stochastic bifurcation. Before proving the main theorem we give some lemmas and definitions.

In this paper, we analyze the behavior of the solutions for a class of stochastic equation with fractional Brownian motion(fBm):

$$dx_t = f(x_t)dt + g(x_t)dB_t^H, \qquad (2.1)$$

where B_t^H is a fractional Brownian motion defined on a complete probability space (Ω, F, P) endowed with the natural filtration $(F_t)_t$ of B_t^H. Notice that several classical normal forms can be reformulated in this setting and our general results will be applied to the cases of the Transcritical, pitchfork or Hopf bifurcations in section 3. We assume that there exists a singular poin x^* such that $f(x^*) = g(x^*) = 0$. The system has a trivial solution distributed as a Dirac measure localized at the singular point x^*, and we shall address the stochastic bifurcation of these solutions. In contrast to the case of ODEs there are at least five different notions of stochastic bifurcation[12]

Definition 1 (D-Bifurcation) Dynamical bifurcation is concerned with a family random dynamical systems which is differential and has invariant measure μ_α. If there exist a constant α_D satisfying in any neighborhood of α_D, there exist another constant α and the corresponding invariant measure $\nu_\alpha \neq \mu_\alpha$ satisfying $\nu_\alpha \to \mu_\alpha$ as $\alpha \to \alpha_D$. Then the constant α_D is a point of dynamical bifurcation.

Definition 2 (P-Bifurcation) Phenomenological bifurcation is concerned with the change in the shape of density (stationary probability density) of a family random dynamical systems as the change of the parameter. If there exist a constant α_0 satisfying in any neighborhood of α_D, there exist other two constant α_1, α_2 and their corresponding stationary density function $p_{\alpha_1}, p_{\alpha_2}$ satisfying p_{α_1} and p_{α_2} are not equivalent. Then the constant α_0 is a point of phenomenological bifurcation.

Definition 3 (Stochastic Transcritical Bifurcation) A family of RDS $(\varphi_\alpha)_{\alpha\in\Re}$ on $\Re$ undergoes a Stochastic Transcritical Bifurcation at $\alpha = 0$, if

(i) for $\alpha < 0$, φ_α has exactly two ergodic invariant measure in $I(\varphi_\alpha)$: δ_0 which is stable, and $\mu_\alpha = \delta_{a_\alpha}$ with a random variable $a_\alpha < 0 P_{-a.s.}$, which is unstable, and $a_\alpha \to 0$ in probability as $\alpha \uparrow 0$,

(ii) for $\alpha = 0$, δ_0 is only invariant measure and the Lyapunov exponent of φ_0 with respect to δ_0 vanishes,

(iii) for $\alpha > 0$, φ_α has exactly two ergodic invariant measure in $I(\varphi_\alpha)$: δ_0 which is unstable, and $\mu_\alpha = \delta_{a_\alpha}$ with a random variable $a_\alpha > 0 P_{-a.s.}$, which is stable, and $a_\alpha \to 0$ in probability as $\alpha \downarrow 0$,

Definition 4 (Stochastic Pitchfork Bifurcation) A family of RDS $(\varphi_\alpha)_{\alpha\in\Re}$ on $\Re$ undergoes a Stochastic Pitchfork Bifurcation at $\alpha = 0$, if

(i) for $\alpha \le 0$, δ_0 is only invariant measure of φ_α, which is stable for $\alpha < 0$ and the Lyapunov exponent of φ_0 with respect to δ_0 vanishes;

(ii) for $\alpha > 0$ the system possesses besides δ_0, which is unstable, exactly two more ergodic invariant measure μ_α^1, μ_α^2 in $I(\varphi_\alpha)$, described by $\mu_\alpha^i - \delta_{a_\alpha^i}$, $i = 1, 2,$ with random variable $a_\alpha^1 > 0$, $a_\alpha^2 < 0 P_{-a.s.}$ and μ_α^i, $i = 1, 2,$ are stable;

(iii) We have $a_\alpha^i \to 0$ in probability as $\alpha \downarrow 0$, $i = 1, 2.$

Definition 5 (Stochastic Hopf Bifurcation)

In the viewpoint of phenomenological bifurcation:

The stationary solution of the FPK equation which is corresponded to the stochastic differential equation changes from one peak into crater.

In the viewpoint of dynamical bifurcation:

(i) If one of the invariant measures of the stochastic differential equation loses its stability and becomes unstable (i.e. two Lyapunov exponent are positive), moreover the rotation number is not zero. Meanwhile there at least appear one new stable invariant measure.

(ii) The global attractors of the stochastic differential equation change from a single point set into a random topological disk (the closure of the unstable manifold of the unstable invariant measure).

If the stochastic bifurcation of a stochastic differential equation has the above characters, then the stochastic differential equation admits stochastic Hopf bifurcation.

Definition 6 (Stochastically Stable) The trivial solution $x(t, t_0, x_0)$ of stochastic differential equation is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $\alpha > 0$, there exists a $\delta = \delta(\varepsilon, \delta, t_0) > 0$ such that

$$P\{|x(t, t_0, x_0)| < \alpha \text{ for all } t \ge t_0\} \ge 1 - \varepsilon,$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

III. STOCHASTIC STBILITY AND BIFURCATION

Now that we characterized in detail the solutions of equation (2.1), we are in a position address the dynamics of three prominent bifurcations occurring in the study of random dynamical systems: the transcritical, pitchfork and Hopf bifurcations with power diffusion coefficients.

A. Stochastic Transcritical Bifurcation

In this section we first study the dynamics of an SDE with fractional Brownian motion(fBm) given by the normal form of the transcritical bifurcation, before addressing the universality of these behaviors.

We consider the stochastic differential equation with fBm

$$dx_t = (\alpha x_t - x_t^2)dt + \sigma x_t^k dB_t^H$$

(3.1)

Theorem 3.1 For $k=1$, let us denote by x_0 the stationary solution. We have:

(1) When $0 < H < 1/2$ and:

(i)For $\alpha < 0$, the fixed point x_0 of equation (3.1) is stable in probability and the two invariant measures are both $F_{-\infty}^{0}$ measurable;

(ii) For $\alpha > 0$, the fixed point x_0 of equation (3.1) is unstable in probability and the invariant measure is both $F_{-\infty}^{0}$ measurable;

(2) When $H = 1/2$, and:

(i) For $\alpha - \frac{\sigma^2}{2} < 0$, the fi xed point x_0 of equation (3.1) is stable in probability and the two invariant measures are both $F_{-\infty}^{0}$ measurable;

(ii) For $\alpha - \frac{\sigma^2}{2} > 0$, the fi xed point x_0 of equation (3.1) is unstable in probability and the invariant measure is both $F_{-\infty}^{0}$ measurable;

(3) When $1/2 < H < 1$, and $\alpha \in \Re$, the fi xed point x_0 of equation (3.1) is almost surely exponentially stable. Moreover, any solution almost surely reaches zero in finite time.

Proof. We now calculate the Lyapunov exponent for each of these measures of the SDE(3.1). The linearized RDS $v_t = D\varphi(t,\omega,x)v$ satisfies the linearized SDE(3.1) with fBm

$$dv_t = [\alpha - \sigma^2 Ht^{2H-1} - 2(\varphi(t,\omega)x)]v_t dt + \sigma v_t dB_t^H,$$

hence

$$D\varphi(t,\omega,x)v = v\exp\left(\alpha t - \frac{\sigma^2}{2}t^{2H} + \sigma B_t^H(\omega) - 2\int_0^t(\varphi(s,\omega)x)ds\right).$$

Thus, if $\mu_\omega = \delta_{x_0(\omega)}$ is a φ-invariant measure its Lyapunov exponent is

$$\lambda_{(\mu)} = \lim_{t\to\infty}\frac{1}{t}\log\left\|D_\varphi(t,\omega,x)v\right\|$$

$$= \alpha - \frac{\sigma^2}{2}t^{2H-1} - 2\lim_{t\to\infty}\frac{1}{t}\int_0^t(\varphi(t,\omega)x)^{N-1}ds$$

$$= \alpha - \frac{\sigma^2}{2}\lim_{t\to\infty}t^{2H-1} - 2\mathrm{E}x_0 \mathrm{x}$$

provided the IC $x_0 \in L^1(\mathrm{P})$ is satisfied.

From SDE(3.1) with fBm, we can obtain the solution of SDE(3.1) which is explicitly given by which is solved by

$$\varphi_\alpha(t,\omega)x = \frac{xe^{\alpha t - \frac{\sigma^2}{2}t^{2H} + \sigma B_t^H(\omega)}}{1 + x\int_0^t e^{\alpha s - \frac{\sigma^2}{2}s^{2H} + \sigma B_s^H(\omega)}ds}.$$

We now determine the domain $D_\alpha(t,\omega)$ and the range $R_\alpha(t,\omega)$ of $\varphi_\alpha(t,\omega)$: $D_\alpha(t,\omega) \to R_\alpha(t,\omega)$

We have

$$D_\alpha(t,\omega) = \begin{cases} (-d_\alpha(t,\omega),\infty), & t>0, \\ \Re, & t=0, \\ (-\infty, d_\alpha(t,\omega)), & t<0, \end{cases}$$

where

$$d_\alpha(t,\omega) = \frac{1}{\left(\left|\int_0^t e^{\alpha s - \frac{\sigma^2}{2}s^{2H} + \sigma B_s^H(\omega)}ds\right|\right)} > 0,$$

and

$$R_\alpha(t,\omega) = D_\alpha(-t,\theta(t)\omega) = \begin{cases} (-\infty, r_\alpha(t,\omega)), & t>0, \\ \Re, & t=0, \\ (-r_\alpha(t,\omega),\infty), & t>0, \end{cases}$$

where

$$r_\alpha(t,\omega)=d_\alpha(-t,\theta(t)\omega)$$
$$=\frac{e^{\alpha t-\frac{\sigma^2}{2}t^{2H}+\sigma B_t^H}}{\left(\left|\int_0^t e^{(\alpha s-\frac{\sigma^2}{2}s^{2H}+\sigma B_t^H(\omega))}ds\right|\right)}>0.$$

We can now determine

(i) For $0<H<1/2$,

$$E_\alpha=\begin{cases}[0,d_\alpha^-(\omega)], & \alpha>0,\\ 0, & \alpha=0,\\ (-d_\alpha^+(\omega),0], & \alpha<0,\end{cases}$$

(ii) For $H=1/2$,

$$E_\alpha=\begin{cases}[0,d_\alpha^-(\omega)], & \alpha>\frac{\sigma^2}{2},\\ 0, & \alpha=\frac{\sigma^2}{2},\\ (-d_\alpha^+(\omega),0], & \alpha<\frac{\sigma^2}{2},\end{cases}$$

(iii) For $1/2<H<1$,

$$E_\alpha=(-d_\alpha^+(\omega),0],\qquad \alpha,\sigma\in\Re,$$

where

$$0<d_\alpha^\pm(\omega)=\frac{1}{\left(\left|\int_0^{\pm\infty} e^{(\alpha s-\frac{\sigma^2}{2}s^{2H}+\sigma B_t^H(\omega))}ds\right|\right)}<\infty.$$

(1)For $0<H<1/2$ and:

(i) When $\alpha\in\Re$, the inclusion closed(IC) $\mu_{1,\omega}^\alpha=\delta_0$ is trivially satisfied and we obtain

$$\lambda(\mu_1^\alpha)=\alpha$$

so μ_1^α is stable for $\alpha<0$ and unstable for $\alpha>0$.

(ii) When $\alpha>0$, $\mu_{2,\omega}^\alpha=\delta_{d_-^\alpha(\omega)}$ is $F_{-\infty}^{0}$ measurable, hence the density p^α of $\rho^\alpha=\mathrm{E}_{\mu_2^\alpha}$ satisfies the Fokker-Planck-Kolmogorov(FPK) equation[4]

$$\frac{\partial p^\alpha(t,x)}{\partial t}+\frac{\partial((\alpha x-\frac{\sigma^2}{2}x-x^2)p^\alpha(t,x))}{\partial x}$$
$$-\frac{\partial^2((x\int_0^t x\phi(s,t)ds)p^\alpha(t,x))}{\partial x^2}=0$$

which has the unique probability density solution $p^\alpha(t,x)$ [4].

Since

$$\mathrm{E}_{\mu_2^\alpha}x=\mathrm{E}(d_-^\alpha)=\int_0^\infty xp^\alpha(x)ds<\infty,$$

the IC is satisfi ed. The calculation of the Laypunov exponent is accomplished by observing that

$$d_-^\alpha(\theta_t\omega)=\frac{e^{(\alpha t-\frac{\sigma^2}{2}t^{2H}+\sigma B_t^H(\omega))}}{\int_{-\infty}^t(e^{(\alpha s-\frac{\sigma^2}{2}s^{2H}+\sigma B_s^H(\omega))ds}}\ \frac{\psi'(t)}{\psi(t)}$$

where

$$\psi(t)=\int_{-\infty}^t e^{(\alpha s-\frac{\sigma^2}{2}s^{2H}+\sigma B_s^H(\omega))}ds..$$

Hence by the ergodic theorem

$$\mathrm{E}(d_-^\alpha)=\lim_{t\to\infty}\frac{1}{t}\log\psi(t)=\alpha,$$

finally

$$\lambda(\mu_2^\alpha)=-\alpha<0.$$

(iii) When $\alpha<0$, $\mu_{3,\omega}^\alpha=\delta_{-d_+^\alpha(\omega)}$ is $F_{-\infty}^{0}$ measurable. Since $L\left(d_+^\alpha\right)=L\left(d_-^{-\alpha}\right)$ and N is even,

$$\mathrm{E}(-d_+^\alpha)=-\mathrm{E}(d_-^{-\alpha})=\alpha$$

thus

$$\lambda(\mu_3^\alpha)=-\alpha>0.$$

Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $\alpha_D=\frac{\sigma^2}{2}$ and a P-bifurcation of the $\nu_\pm^\alpha$ at $\alpha p.$ Then the SDE(3.1) with fBm admits stochastic Transcritical bifurcation.

(2) $H=1/2$ and:

(i) When $\alpha,\sigma\in\Re$, the inclusion closed(IC) for $\mu_{1,\omega}^\alpha=\delta_0$ is trivially satisfied and we obtain

$$\lambda(\mu_1^\alpha)=\alpha-\frac{\sigma^2}{2}$$

so μ_1^α is stable for $\alpha-\frac{\sigma^2}{2}<0$ and unstable for $\alpha-\frac{\sigma^2}{2}>0.$

(ii)When $\alpha-\frac{\sigma^2}{2}>0$, $\mu^{\alpha}_{2,\omega}=\delta_{d^{\alpha}_{-}(\omega)}$ is $F^{0}_{-\infty}$ measurable, hence the density p^{α} of $\rho^{\alpha}=\mathrm{E}\mu^{\alpha}_{2}$ satisfies the Fokker-Planck-Kolmogorov(FPK) equation[4]

$$\frac{\partial p^{\alpha}}{\partial t}=-\frac{\partial((\alpha x-\frac{\sigma^2}{2}x-x^2)p^{\alpha})}{\partial x}+\frac{\partial^2(x^2p^{\alpha})}{\partial x^2}=0 \quad (3.2)$$

which has the unique probability density solution

$$p^{\alpha}=N_{\alpha}x^{2\alpha/\sigma^2-1}\exp(-\frac{2x}{\sigma^2}),\quad x>0,$$

where

$$N_{\alpha}^{-1}=\Gamma\left(\frac{2\alpha}{\sigma^2}\right)\left(\frac{\sigma^2}{2}\right).$$

Since

$$\mathrm{E}_{\mu^{\alpha}_{2}}x=\mathrm{E}(d^{\alpha}_{-})=\int_0^{\infty}xp^{\alpha}(x)dx<\infty,$$

the IC is satisfied. The calculation of the Laypunov exponent is accomplished by observing that

$$d^{\alpha}_{-}(\theta_t\omega)=\frac{e^{(\alpha t-\frac{\sigma^2}{2}t^{2H}+\sigma B^H_t(\omega))}}{\int_{-\infty}^{t}(e^{(\alpha s-\frac{\sigma^2}{2}s^{2H}+\sigma B^H_s(\omega))ds}}\frac{\psi'(t)}{\psi(t)}$$

where

$$\psi(t)=\int_{-\infty}^{t}e^{(\alpha s-\frac{\sigma^2}{2}s^{2H}+\sigma B^H_s(\omega))}ds.$$

Hence by the ergodic theorem

$$\mathrm{E}(d^{\alpha}_{-})=\lim_{t\to\infty}\frac{1}{t}\log\psi(t)=\alpha-\frac{\sigma^2}{2},$$

finally

$$\lambda(\mu^{\alpha}_{2})=-(\alpha-\frac{\sigma^2}{2})<0.$$

(iii) When $\alpha-\frac{\sigma^2}{2}<0$, $\mu^{\alpha}_{3,\omega}=\delta_{-d^{\alpha}_{+}(\omega)}$ is $F^{0}_{-\infty}$ measurable.

Since $L\left(d^{\alpha}_{+}\right)=L\left(d^{-\alpha}_{-}\right)$,

$$\mathrm{E}(-d^{\alpha}_{+})=-\mathrm{E}(d^{-\alpha}_{-})=\alpha-\frac{\sigma^2}{2}$$

thus

$$\lambda(\mu^{\alpha}_{3})=-(\alpha-\frac{\sigma^2}{2})>0.$$

Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $\alpha_D=\frac{\delta^2}{2}$ and a P-bifurcation of the $\nu^{\alpha}_{\pm}$ at $\alpha p=\frac{\sigma^2}{2}$. Then the SDE(3.1) with fBm admits stochastic Transcritical bifurcation.

(3) $1/2<H<1$, the inclusion closed(IC) for $\mu^{\alpha}_{1,\omega}=\delta_0$ is trivially satisfied and we obtain

$$\lambda(\mu^{\alpha}_{1,2,3})=-\infty,$$

so μ^{α}_{1}, μ^{α}_{2} and μ^{α}_{3} are stable for $\alpha\in\Re$.

Let us now further describe the dynamics of the solutions as a function of H.

- Case $H=1/2$: We observe that for any $\alpha>0$, the unstable deterministic fixed point becomes asymptotically exponentially stable when the noise parameter σ is large enough. For $\alpha>\frac{\sigma^2}{2}$, two symmetrical stationary distributions appear. For $\frac{\sigma^2}{2}<\alpha<\sigma^2$ the stationary distribution concentrates at zero and has a non increasing density diverging at zero. For $\alpha\geq\sigma^2$, the probability density function of the stationary distribution vanishes at zero and has a unique maximum reached for $x\pm\sqrt{\alpha-\sigma^2}$. There is hence a qualitative transition at $\alpha=\sigma^2$, or P-bifurcation. In comparison with the deterministic bifurcation, the loss of stability is delayed and noise tends to stabilize the saddle point.
- Case $0<H<1/2$: the deterministic picture is qualitatively and quantitatively recovered: for $\alpha<0$, x_0 is stable in probability and is the unique stationary solution, and for $\alpha>0$ is unstable in probability, two additional stationary solutions appear.
- Case $1/2<H<1$: the solution x_0 is always stable in probability. This result can appear relatively surprising at first sight. Indeed, in the deterministic case, α is the exponential rate of divergence from the solution 0. However, adding a (possibly small) diffusion term proportional to $|x|dB^H(t)$ with $1/2<H<1$ stabilizes x_0 in probability whatever the value of the noise intensity. This observation, added to the fact that the solution is not exponentially asymptotically stable (though stable in probability) raises the question of how this convergence occurs. Starting from a positive initial condition, we observe that the solution is evolving in the half-plane $x>0$ and does not show any extinction clue. However, it suddenly reaches zero where it is absorbed after that random transient phase. This perfectly illustrates a typical behavior of the solutions of the transcritical equation for $1/2<H<1$: the extinction time was shown to be almost surely finite. Moreover, trajectories that did not hit zero are distributed according to a quasi-stationary distributions as long as $1/2<H<1$. Interestingly, for the first time it is found that the Hurst parameter affects both bifurcation conclusions and large deviations which is significantly different from the classical Brownian motion process. This fact is due to the long-range dependence (LRD) property of the fBm.

Theorem 3.2 For $k<1, 0<H<1$, then the fixed point x_0 of equation (3.1) is asymptotically stable in probability whatever $\alpha \in \Re$ and $\sigma \neq 0$.

Proof. Case $0<H<0.5$: Let us consider $V(x)=x^k$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$L\,V \leq kx^{k-1}(\alpha x - x^2) - k(1-k)\sigma^2 Ht^{2H-1}x^{3k-2}.$$

Since we assume that $0<H<0.5$ and $-k(1-k)Ht^{2H-1}x^{3k-2}$ at 0, the leading term close to 0 is of order $-x^{k+1}$ which is strictly negative for sufficiently small x. More precisely, there exists r such that $V \in C_2^0(U_r)$ where U_r is the open ball of radius r and such that $L\,V<0$ for all $x \in U_r$. Moreover, for any ε such that $0<\varepsilon<r$ we have $V(r)$ and $L\,V < -c_\varepsilon < 0$ for all $r>x>\varepsilon$. Theorem of [21] therefore applies and concludes the proof of the stability in probability of the solution x_0 whatever the parameters $\alpha \in \Re, \sigma>0$.

Case $0.5 \leq H < 1$: Let us consider $V(x)=x^k$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$L\,V \leq kx^{k-1}(\alpha x - x^2) - k(1-k)\sigma^2 Ht^{2H-1}x^{3k-2}.$$

Since we assume that $0<H<0.5$ and $-kx^{k+1}$ at 0, the leading term close to 0 is of order $-k(1-k)\sigma^2 Ht^{2H-1}x^{3k-2}$ which is strictly negative for sufficiently small x. More precisely, there exists r such that $V \in C_2^0(U_r)$ where U_r is the open ball of radius r and such that $L\,V<0$ for all $x \in U_r$. Moreover, for any ε such that $0<\varepsilon<r$ we have $V(r)$ and $L\,V < -c_\varepsilon < 0$ for all $r>x>\varepsilon$. Theorem of [21] therefore applies and concludes the proof of the stability in probability of the solution x_0 whatever the parameters $\alpha \in \Re, \sigma>0$. When $H=0.5$, then $k \in [1/2, 1)$.

Theorem 3.3 For $k>1$, let us denote by x_0 the stationary solution $X=0$ a.s. for all t. We have:

(1)For $0<H\leq 1/2$ and:

(i) When $\alpha<0$, the fixed point x_0 of equation (3.1) is stable in probability and the two invariant measures are both $F_{-\infty}^{\ 0}$ measurable;

(ii) When $\alpha>0$, the fixed point x_0 of equation (3.1) is unstable in probability and the invariant measure is both $F_0^{\ \infty}$ measurable;

(2) For $1/2<H<1$ and $\alpha \in \Re$, the fixed point x_0 of equation (3.1) is not stable in probability.

Proof. **Case** $0<H\leq 1/2$: Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$L\,V \leq 2x(\alpha x - x^2) + \sigma^2 Ht^{2H-1}x^{2k}.$$

Since we assume that $0<H\leq 1/2$ and $-2x^3$ at 0, the term $L\,V$ is equivalent to $2\alpha x^2$ and hence locally has the sign of α. For $\alpha<0$, we can directly apply theorem [21]. For $\alpha>0$, we use $V(x)=-\log(x)$ again and conclude that 0 is not stable in probability.

Case $0.5<H\leq 1$: Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$L\,V \leq 2x(\alpha x - x^2) + \sigma^2 Ht^{2H-1}x^{2k}.$$

Since we assume that $1/2<H<1$ and $-2x^3$ at 0, the term $L\,V$ is equivalent to $\sigma^2 Ht^{2H-1}x^4$ and hence $L\,V>0$, we can directly apply theorem [21]. For $\alpha \in \Re$ we conclude that 0 is not stable in probability.

B. Stochastic Pitchfork Bifurcation

In this section we first study the dynamics of an SDE with fractional Brownian motion(fBm) given by the normal form of the pitchfork bifurcation, before addressing the universality of these behaviors.

We consider the supercritical stochastic pitchfork bifurcation with fBm

$$dx_t = (\alpha x_t - x_t^3)dt + \sigma |x_t|^k dB_t^H \tag{3.3}$$

Theorem 3.4 For $H=1/2$, let us denote by x_0 the stationary solution $X=0$ a.s. for all t. We have:

(1)For $k=1$ and:

(i) When $\alpha - \frac{\sigma^2}{2} \leq 0$, the fixed point x_0 of equation (3.1) is stable in probability and the only invariant measure is $\mu_\omega^\alpha = \delta_0$;

(ii) When $\alpha - \frac{\sigma^2}{2} > 0$, the fixed point x_0 of equation (3.1) is stable in probability and the three forward Markov measure $\mu_\omega^\alpha = \delta_0$ and $\nu_{\pm,\omega}^\alpha = \delta_{d_\alpha(\omega)}$, where

$$d_\alpha(\omega) := \left(2\int_{-\infty}^{0} e^{2\alpha t - \sigma^2 t + \sigma B_t^H(\omega) dt}\right)^{-\frac{1}{2}}.$$

(2)For $k<1$, the fixed point x_0 of equation (3.1) is asymptotically stable in probability whatever $\alpha \in \Re$ and $\sigma \neq 0$.

(3) For $k>1$, x_0 is

–stable in probability for $\alpha<0$

–unstable in probability for $\alpha>0$

Proof. (1) **Case k=1**.

We now calculate the Lyapunov exponent for each of these measures of the SDE(3.3). The linearized RDS $v_t = D_\varphi(t,\omega,x)v$ satisfies the linearized SDE(3.3) with fBm

$$dv_t = (\alpha - \sigma^2 H t^{2H-1} - 3(\varphi(t,\omega)x)^2 v_t dt + \sigma v_t dB_t^H, \quad (3.4)$$

hence

$$D\varphi(t,\omega,x)v = v\exp\left(\alpha t - \frac{\sigma^2}{2}t^{2H} + \sigma B_t^H(\omega) - 3\int_0^t (\varphi(s,\omega)x))^2 ds\right).$$

Thus, if $\mu_\omega = \delta_{x_0(\omega)}$ is a φ-invariant measure its Lyapunov exponent is

$$\lambda(\mu) = \lim_{t\to\infty}\frac{1}{t}\log\|D\varphi(t,\omega,x)v\|$$
$$= \alpha - \frac{\sigma^2}{2}t^{2H-1} - 2\lim_{t\to\infty}\frac{1}{t}\int_0^t \left((\varphi(t,\omega)x)^{N-1}\right)ds$$
$$= \alpha - \frac{\sigma^2}{2} - 3\mathrm{E}x_0^2$$

provided the IC $x_0 \in L^1(\mathrm{P})$ is satisfied.

From SDE(3.1) with fBm, we can obtain the solution of SDE(3.1) which is explicitly given by which is solved by

$$\varphi_\alpha(t,\omega)x = \frac{xe^{\alpha t - \frac{\sigma^2}{2}t + \sigma B_t^H(\omega)}}{\left(1 + 2x^2\int_0^t e^{2(\alpha s - \frac{\sigma^2}{2}s + \sigma B_s^H(\omega))} ds\right)^{1/2}}.$$

We now determine the domain $D_\alpha(t,\omega)$ and the range $R_\alpha(t,\omega)$ of $\varphi_\alpha(t,\omega)$: $D_\alpha(t,\omega) \to R_\alpha(t,\omega)$

We have

$$D_\alpha(t,\omega) = \begin{cases} (-d_\alpha(t,\omega),\infty), & t>0, \\ \Re, & t=0, \\ (-\infty, d_\alpha(t,\omega)), & t<0, \end{cases}$$

where

$$d_\alpha(t,\omega) = \frac{1}{\left(2\left|\int_0^t e^{(\alpha s - \frac{\sigma^2}{2}s + \sigma B_s^H(\omega))ds}\right|\right)^{1/2}} > 0,$$

and

$$R_\alpha(t,\omega) = D_\alpha(-t,\theta(t)\omega) = \begin{cases} (-\infty, r_\alpha(t,\omega)), & t>0, \\ \Re, & t=0, \\ (-r_\alpha(t,\omega),\infty), & t>0, \end{cases}$$

where

$$r_\alpha(t,\omega) = d_\alpha(-t,\theta(t)\omega) = \frac{e^{\alpha t - \frac{\sigma^2}{2}t + \sigma B_t^H}}{\left(2\left|\int_0^t e^{\left(\alpha s - \frac{\sigma^2}{2}s + \sigma B_t^H(\omega)\right)} ds\right|\right)^{1/2}} > 0.$$

We can now determine

$$E_\alpha = \begin{cases} \left[0, d_\alpha^-(\omega)\right], & \alpha > \frac{\sigma^2}{2}, \\ 0, & \alpha = \frac{\sigma^2}{2}, \\ (-d_\alpha^+(\omega), 0], & \alpha < \frac{\sigma^2}{2}, \end{cases}$$

where

$$0 < d_\alpha^\pm(\omega) = \frac{1}{\left(2\left|\int_0^{\pm\infty} e^{\left(\alpha s - \frac{\sigma^2}{2}s + \sigma B_t^H(\omega)\right)} ds\right|\right)^{1/2}} < \infty.$$

From Proposition 2.1, there are ergodic invariant measures of SDE(3.1) with fBm as follows:

(i) For $\alpha - \frac{\sigma^2}{2} \le 0$, the only invariant measure is $\mu_\omega^\alpha = \delta_0$;

(ii) For $\alpha - \frac{\sigma^2}{2} > 0$, the fixed point x_0 of equation (3.1) is stable in probability and the three forward Markov measure $\mu_\omega^\alpha = \delta_0$ and $\nu_{\pm,\omega}^\alpha = \delta_{d_\alpha(\omega)}$, where

$$d_\alpha(\omega) := \left(2\int_{-\infty}^0 e^{2\alpha t - \sigma^2 t + \sigma B_t^H(\omega)} dt\right)^{-\frac{1}{2}}.$$

We have $\mathrm{E}d_\alpha^2(\omega)=\alpha-\frac{\sigma^2}{2}$. Solving the forward Fokker-Planck-Kolmogorov(FPK) equation[12]

$$L^* p^\alpha = -\frac{\partial\left(\left(\alpha x-\frac{\sigma^2}{2}x-x^3\right)p^\alpha\right)}{\partial x}+\frac{\partial^2((x^2p^\alpha))}{\partial x^2}=0$$

yields

(i) $p_\alpha=\delta_0$ for all α, σ,

(ii) for $\alpha-\frac{\sigma^2}{2}>0$

$$q_\alpha^+=N_\alpha x^{\frac{2\alpha}{\sigma^2}-1}\exp\left(-\frac{2\alpha}{\sigma^2}x\right),\quad x>0,$$

$$q_\alpha^-(x)=q_\alpha^+(-x).$$

Naturally the invariant measure $\nu_{\pm,\omega}^\alpha=\delta_{d_\alpha(\omega)}$ are those corresponding to the stationary measures $q_\alpha^\pm$, Hence all invariant measure are Markov measures.

The two families of densities $\left(q_\alpha^\pm\right)_{\alpha-\frac{\sigma^2}{2}>0}$ clearly undergo a P-bifurcation at the parameter value $\alpha p=\frac{\sigma^2}{2}$.

By Proposition 2.1, the Lyapunov exponent of the linearized SDE(3.3) with respect to the three measure is

(i) for $\mu^\alpha=\delta_0,\lambda\left(\mu^\alpha\right)=\alpha-\frac{\sigma^2}{2}$,

(ii)for $\nu_{\pm,\omega}^\alpha=\delta_{d_\alpha(\omega)}$,

$$\lambda\left(\nu^\alpha\right)=\alpha-\frac{\sigma^2}{2}-3\mathrm{E}d_\alpha^2=-2\alpha+\sigma^2.$$

Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $\alpha_D=\frac{\sigma^2}{2}$ and a P-bifurcation of the $\nu_\pm^\alpha$ at $\alpha_P=\frac{\sigma^2}{2}$. Then the SDE(3.4) with fBm admits stochastic pitchfork bifurcation.

(2)**Case k**< 1.

Let us consider $V(x)=x^k$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at X = 0 . Moreover, we have:

$$L\,V\le kx^{k-1}(\alpha x-x^3)-\frac{k(1-k)}{2}\sigma^2x^{3k-2}.$$

Since we assume that $k<1$ and $-kx^{k+2}$ at 0,the leading term close to 0 is of order $-k(1-k)\sigma^2x^{3k-2}/2$ which is strictly negative for sufficiently small x. More precisely, there exists $r>0$ such that $V\in C_2^0(U_r)$ where U_r is the open ball of radius r and such that $L\,V<0$ for all $x\in U_r$ .Moreover, for any ε such that $0<\varepsilon<r$ we have $V(r)$ and $L\,V<-c_\varepsilon$ <0 for all $r>x>\varepsilon$ .Theorem of [21] therefore applies and concludes the proof of the stablility in probability of the solution x_0 whatever the parameters $\alpha\in\Re$, $\sigma>0$ and $k\in[1/2,1)$.

(3)Case k> 1

Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at X = 0 . Moreover, we have:

$$L\,V\le 2x\left(\alpha x-x^3\right)+\frac{\sigma^2}{2}x^{2k}.$$

Since we assume that $k>1$ and $-2x^4$ at 0,the term $L\,V$ is equivalent to $2\alpha x^2$ and hence locally has the sign of α. For $\alpha<0$, we can directly apply theorem of [21]. For $\alpha>0$,we use $V(x)=-\log(x)$ again and conclude that 0 is not stable in probability.

Theorem 3.5 For $0<H<1/2$, let us denote by x_0 the stationary solution $X=0$ a.s. for all t. We have:

(1)For $k=1$ and:

(i) $\alpha\le 0$, the fixed point x_0 of equation (3.1) is stable in probability and the only invariant measure is $\mu_\omega^\alpha=\delta_0$;

(ii) $\alpha>0$, the fixed point x_0 of equation (3.1) is stable in probability and the three forward Markov measure and $\mu_\omega^\alpha=\delta_0$ and $\nu_{\pm,\ \omega}^\alpha=\delta_{d_\alpha(\omega)}$, where

(2)For $k<1$ and:

(i) $\alpha<0$, x_0 is stable in probability and no other stationary solution exist.

(ii) $\alpha>0$, x_0 is unstable in probability for $\alpha>0$ and stationary solution exist.

(3)For $k>1$ and:

(i) $\alpha<0$, x_0 is stable in probability and no other stationary solution exist.

(ii) $\alpha>0$, x_0 is unstable in probability for $\alpha>0$ and stationary solution exist.

Proof. (1) **Case k**=1

From SDE(3.3) with fBm, we have the linearized RDS v_t $=D\varphi(t,\omega,x)v$ satisfies the linearized SDE(3.3) with fBm

$$dv_t=\left(\alpha-\sigma^2Ht^{2H-1}-3\left(\phi(t,\omega)x\right)^2\right)v_tdt \\ +\sigma v_t dB_t^H, \tag{3.5}$$

hence

$$D\phi(t,\omega,x)v=v\exp\left(\alpha t-\frac{\sigma^2}{2}t^{2H}+\sigma B_t^H(\omega)-3\int_0^t\left(\varphi(s,\omega)x\right)^2ds\right).$$

Thus, if $\mu_\omega=\delta_{x_0(\omega)}$ is a φ-invariant measure its Lyapunov exponent is

$$\lambda(\mu)=\lim_{t\to\infty}\frac{1}{t}\log\left\|D\varphi(t,\omega,x)v\right\|$$
$$=\alpha-\frac{\sigma^2}{2}t^{2H-1}-2\lim_{t\to\infty}\frac{1}{t}\int_0^t(\varphi(t,\omega)x)^{N-1}ds$$
$$=\alpha-\frac{\sigma^2}{2}t^{2H-1}-3\mathrm{E}x_0^2$$

provided the IC $x_0\in L^1(\mathrm{P})$ is satisfied.

From SDE (3.6) with fBm, we can obtain the solution of SDE(3.6) which is explicitly given by which is solved by

$$\varphi_\alpha(t,\omega)x=\frac{xe^{\alpha t-\frac{\sigma^2}{2}t^{2H-1}+\sigma B_t^H(\omega)}}{\left(1+2x^2\int_0^t e^{2\left(\alpha s-\frac{\sigma^2}{2}s^{2H-1}+\sigma B_s^H(\omega)\right)}ds\right)^{1/2}}.$$

From Proposition 2.1, there are ergodic invariant measures of SDE (3.6) with fBm as follows:

(i) For $\alpha\le 0$, the only invariant measure is $\mu_\omega^\alpha=\delta_0$;

(ii) For $\alpha>0$, the fixed point x_0 of equation (3.1) is stable in probability and the three forward Markov measure $\mu_\omega^\alpha=\delta_0$ and $v_{\pm,\omega}^\alpha=\delta_{d_\alpha(\omega)}$, where

$$d_\alpha(\omega):=\left(2\int_{-\infty}^0 e^{2\alpha t-\sigma^2 t^{2H-1}+\sigma B_t^H(\omega)}dt\right)^{-\frac{1}{2}}.$$

We have $\mathrm{E}d_\alpha^2(\omega)=\alpha-\frac{\sigma^2}{2}$. Solving the forward Fokker-Planck-Kolmogorov(FPK) equation[4]

$$\frac{p^\alpha(t,x)}{\partial t}+\frac{\partial\left((\alpha x-x^3)p^\alpha(t,x)\right)}{\partial x}-\frac{\partial^2\left(x^2\sigma^2Ht^{2H-1}p^\alpha(t,x)\right)}{\partial x^2}=0$$

yields

(i) $p_\alpha=\delta_0$ for all α,σ,

(ii) for $\alpha>0$ $q_\alpha^-(x)=q_\alpha^+(-x), x>0$.

Naturally the invariant measure $v_{\pm,\omega}^\alpha=\delta_{d_\alpha(\omega)}$ are those corresponding to the stationary measures $q_\alpha^\pm$, Hence all invariant measure are Markov measures.

The two families of densities $(q_\alpha^\pm)_{\alpha>0}$ clearly undergo a P-bifurcation.

By Proposition 2.1, the Lyapunov exponent of the linearized SDE (3.6) with respect to the three measure is

(i) for $\mu^\alpha=\delta_0,\lambda(\mu^\alpha)=\alpha$,

(ii) for $v_{\pm,\omega}^\alpha=\delta_{d_\alpha(\omega)},\lambda(v^\alpha)=\alpha-3\mathrm{E}d_\alpha^2=-2\alpha.$

Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $\alpha_D=0$ and and a P-bifurcation of the $v_\pm^\alpha$ at $\alpha_p=\frac{\sigma^2}{2}$. Then the SDE (3.4) with fBm admits stochastic pitchfork bifurcation.

(2)Case k< 1

Let us consider $V(x)=x^k$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e .it is twice continuously differentiable except at X = 0 . Moreover, we have:

$$L\,V\le kx^{k-1}(\alpha x-x^3)-k(1-k)\sigma^2Ht^{2H-1}x^{3k-2}.$$

Since we assume that $k<1$ and $-k(1-k)\sigma^2Ht^{2H-1}x^{3k-2}$ at 0, the leading term close to 0 is of order $k\alpha x^k$ which is strictly negative for sufficiently small x and $\alpha<0$;the leading term close to 0 is of order $k\alpha x^k$ which is strictly positive for sufficiently small x and $\alpha>0$, we can directly apply theorem of [2]. For $\alpha<0$, x_0 is stable in probability; for $\alpha>0$, x_0 is stable in probability.

(3)Case k> 1

Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at X = 0 . Moreover, we have:

$$2x(\alpha x-x^3)+2\sigma^2Ht^{2H-1}x^{2k}.$$

Since we assume that $k>1$ and $2\sigma^2Ht^{2H-1}x^{2k}$ at 0, the term $L\,V$ is equivalent to $2\alpha x^2$ and hence locally has the sign of α. For $\alpha<0$,we can directly apply theorem of [21], x_0 is stable in probability. For $\alpha>0$, we use $V(x)=-\log(x)$ again and conclude that x_0 is not stable in probability.

Theorem 3.6 For $1/2<H<1$, , let us denote by x_0 stationary solution X = 0 a.s.for all t. We have:

(1)For $k\le 1$, $X=0$ is asymptotically stable in probability whatever $\alpha\in\Re$ and $\sigma\ne 0$.

(2)For $k>1$, $X=0$ is unstable in probability for $\alpha\in\Re$ and $\sigma\ne 0$.

Proof. (1) **Case k= 1**

From SDE (3.3) with fBm, The linearized RDS $v_t=D_\varphi(t,\omega,x)v$ satisfies the linearized SDE(3.3) with fBm

$$dv_t=\left(\alpha-\sigma^2Ht^{2H-1}-3(\varphi(t,\omega)x)^2\right)v_tdt+\sigma v_t dB_t^H, \tag{3.6}$$

hence

$$D\varphi(t,\omega,x)v = v\exp\left(\alpha t - \frac{\sigma^2}{2}t^{2H} + \sigma B_t^H(\omega) - 3\int_0^t (\varphi(s,\omega)x)^2 ds\right).$$

Thus, if $\mu_\omega = \delta_{x_0(\omega)}$ is a φ -invariant measure its Lyapunov exponent is

$$\lambda(\mu) = \lim_{t\to\infty}\frac{1}{t}\log\|D\varphi(t,\omega,x)v\|$$
$$= \alpha - \frac{\sigma^2}{2}t^{2H-1} - 2\lim_{t\to\infty}\frac{1}{t}\int_0^t (\varphi(t,\omega)x)^{N-1}ds$$
$$= \alpha - \frac{\sigma^2}{2}t^{2H-1} - 3\mathrm{E}x_0^2$$

Since we assume that $k > 1$, then $\lambda(\mu) < 0$ for $\alpha \in \Re$.

(2)Case k< 1

Let us consider $V(x) = x^k$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously dierentiable except at X = 0 . Moreover, we have:

$$L\,V \le kx^{k-1}\left(\alpha x - x^3\right) - k(1-k)\sigma^2 Ht^{2H-1}x^{3k-2}.$$

Since we assume that $k < 1$ and $-kx^{k+2}$ at 0,the leading term close to 0 is of order $-k(1-k)\sigma^2 x^{3k-2}/2$ which is strictly negative for sufficiently small x. More precisely, there exists $r > 0$ such that $V \in C_2^0(U_r)$ where U_r is the open ball of radius r and such that $L\,V < 0$ for all $x \in U_r$ .Moreover, for any ε such that $0 < \varepsilon < r$ we have $V(r)$ and $L\,V < -c_\varepsilon < 0$ for all $r > x > \varepsilon$ .Theorem of [21] therefore applies and concludes the proof of the stablility in probability of the solution x_0 whatever the parameters $\alpha \in \Re$, $\sigma > 0$ and $k \in [1/2, 1)$.

(3)Case k> 1

Let us consider $V(x) = x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at X = 0 . Moreover, we have:

$$2x\left(\alpha x - x^3\right) + 2\sigma^2 Ht^{2H-1}x^{2k}.$$

Since we assume that $k < 1$ and $-kx^{k+2}$ at 0,the leading term close to 0 is of order $2\sigma^2 Ht^{2H-1}x^{2k}$ which is strictly negative for sufficiently small x. More precisely, there exists $r > 0$ such that $V \in C_2^0(U_r)$ where U_r is the open ball of radius r and such that $L\,V > 0$ for all $x \in U_r$ .Moreover, for any ε such that $0 < \varepsilon < r$ we have $V(r)$ and $L\,V > c_\varepsilon >$ 0 for all $r > x > \varepsilon$ .Theorem of [2] therefore applies and concludes the proof of the stablility in probability of the solution x_0 whatever the parameters $\alpha \in \Re$, $\sigma > 0$.

C. Stochastic Hopf Bifurcation

We study in this section the stochastic Hopf bifurcation with multiplicative noise. For simplicity, we will consider that the stochastic perturbations are driven by a single with fractional Brownian motion.

We consider the stochastic Hopf normal form with fBm

$$\begin{cases} dX_t = \left(\alpha X_t - Y_t + \varepsilon X_t\left(X_t^2 + Y_t^2\right)dt\right) \\ \qquad + \left(\sigma_1 X_t - \sigma_2 Y_t\right)|Z_t|^{k-1}dB_t^H, \\ dY_t = \left(\alpha Y_t + X_t + \varepsilon Y_t\left(X_t^2 + Y_t^2\right)dt\right) \\ \qquad + \left(\sigma_2 X_t + \sigma_1 Y_t\right)|Z_t|^{k-1}dB_t^H, \end{cases} \tag{3.7}$$

where α and ε correspond to the parameters of the Hopf bifurcation, $Z_t = X_t + iY_t$ and σ_1 and σ_2 are two parameters globally governing the amplitude of the stochastic perturbation.

The deterministic process Z_t corresponding to $X_t = 0$, $Y_t = 0$ for all $t \ge 0$ is solution of (3.7) and is univocally defined by the fact that the modulus of Z_t is null. We denote by R_t the modulus of Z_t and by R_t its argument

Lemma 1. The modulus of the variable $R_t = |Z_t|$ and the argument θ_t satisfy the equations:

$$\begin{cases} dR_t = \left(\alpha R_t + \sigma_2^2 R_t^{2k-1}Ht^{2H-1} + \varepsilon R_t^3\right)dt + \sigma_1 R_t^k dB_t^H \\ d\theta_t = \left(1 - 2\sigma_1\sigma_2 R_t^{2k-2}Ht^{2H-1}\right)dt + \sigma_2 R_t^{k-1}dB_t^H \end{cases} \tag{3.8}$$

where $R_t = |Z_t| = \sqrt{X_t^2 + Y_t^2}$, θ_t given by the arctan $\dfrac{X_t}{Y_t}$

When $X_t > 0$, added or subtracted π when $X_t < 0$ depending on the sign of Y_t, and $\pm\pi/2$ if $X_t = 0, \pm Y_t$.

Proof. Let (X_t, Y_t) be a solution of the Hopf equations (3.7). We apply fractional $It\hat{o}$ formula to the variable $R_t = \sqrt{X_t^2 + Y_t^2}$ .Then we have

$$dR_t = \frac{1}{R_t}\left[\alpha X_t^2 - X_tY_t + \varepsilon X_t^2\left(X_t^2+Y_t^2\right) + \alpha Y_t^2 + X_tY_t\right.$$
$$\left. + \varepsilon Y_t^2\left(X_t^2+Y_t^2\right)\right]dt + \frac{R_t^{2k-2}}{R_t^3}\left[Y_t^2\left(\sigma_1X_t - \sigma_2Y_t\right)^2 Ht^{2H-1}\right.$$
$$\left. + X_t^2\left(\sigma_2Y_t + \sigma_1X_t\right)^2 Ht^{2H-1}\right]dt - \frac{R_t^{2k-2}}{R_t^3}\left[2X_tY_t\left(\sigma_1X_t - \right.\right.$$
$$\left.\left. -\sigma_2Y_t\right)\left(\sigma_2Y_t+\sigma_1X_t\right)Ht^{2H-1}\right]dt + \frac{R_t^{k-1}}{R_t}\left[X_t\left(\sigma_1X_t - \sigma_2Y_t\right)\right.$$
$$\left. + Y_t\left(\sigma_2Y_t+\sigma_1X_t\right)\right]dB_t^H$$
$$= \left[\frac{1}{R_t}\left(\alpha R_t^2 + \varepsilon R_t^4\right) + \frac{1}{R_t^3}\sigma_2^2 R_t^{4+2k-2}Ht^{2H-1}\right]dt$$
$$+ \frac{1}{R_t}\sigma_1 R_t^{2+k-1}dB_t^H$$
$$= \left[\alpha R_t + \sigma_2^2 R_t^{2k-1}Ht^{2H-1} + \varepsilon R_t^3\right]dt + \sigma_1 R_t^k dB_t^H$$

The argument R_t is given by $\theta_t = \arctan\left(\frac{X_t}{Y_t}\right)$ plus possibly constants depending on the sign of X_t and Y_t.
Applying fractional $It\hat{o}$ formula again yields

$$d\theta_t = \left(1 - 2\sigma_1\sigma_2 R_t^{2k-2}Ht^{2H-1}\right)dt + \sigma_2 R_t^{k-1}dB_t^H$$

which ends the proof of the lemma.

It is important to note that the equation on the modulus is uncoupled of the phase equation on R_t The modulus of $\left(X_t, Y_t\right)$ is therefore solution of a stochastic differential equation of type stochastic pitchfork bifurcation . Moreover, when $k=1$,the equations take the simpler form:

$$\begin{cases} dR_t = \left(\left[\alpha + \sigma_2^2 Ht^{2H-1}\right]R_t + \varepsilon R_t^3\right)dt + \sigma_1 R_t\, dB_t^H \\ d\theta_t = \left(1 - 2\sigma_1\sigma_2 R_t^{2k-2}Ht^{2H-1}\right)dt + \sigma_2 dB_t^H \end{cases}$$

and hence the variable R_t is solution of a stochastic pitchfork bifurcation. By application of theorems 3.4-3.6, it is easy to establish the following:

Theorem 3.7 For $k=1$ and:

(i) $H = 1/2$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $\alpha < \frac{\sigma_1^2+\sigma_2^2}{2}$ and asymptotically stochastically unstable if $\alpha > \frac{\sigma_1^2+\sigma_2^2}{2}$. In that case, there exists a new stochastically stable stationary solution with distribution:

$$P_\alpha(x) = N_\alpha x^{-2\left(1-\frac{\lambda}{\sigma_1^2}\right)} e^{-x^2/\sigma_1^2}\mathrm{L}_{x\geq 0}.$$

The null solution of the subcritical Hopf equations is almost surely exponentially unstable if $\alpha > \frac{\sigma_1^2+\sigma_2^2}{2}$ .It is asymptotically stochastically unstable if $\alpha > -\frac{\sigma_1^2+\sigma_2^2}{2}$ and stochastically stable if $\alpha < -\frac{\sigma_1^2+\sigma_2^2}{2}$

(ii) $0 < H < 1/2$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $\alpha < 0$ and asymptotically stochastically unstable if $\alpha > 0$.

(iii) $1/2 < H < 1$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $\sigma_1^2 > \sigma_2^2$ and asymptotically stochastically unstable if $\sigma_1^2 < \sigma_2^2$.

Remark. Let us emphasize the fact that the case $0 < H < 0.5$ is substantially different from the general case: indeed, we had seen that the singular point was always stochastically stable. Surprisingly here the stability of zero depends on the relative values of the real and imaginary parts of the noise.

Theorem 3.8 For $k<1$ .In the case of the supercritical $(\varepsilon = -1)$ stochastic Hopf bifurcation, we have:

(i) $H = 1/2$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $|\sigma_1| < |\sigma_2|$ and asymptotically stochastically unstable if $|\sigma_2| > |\sigma_1|$.

(ii) $0 < H < 1/2$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $\alpha < 0$ and asymptotically stochastically unstable if $\alpha > 0$.

(iii) $1/2 < H < 1$, the null solution of the supercritical Hopf equations is almost surely exponentially stable whatever the parameters $\alpha \in \Re$.

Proof. (1) The case $0 < H < 0.5$, falls in the general analysis developed in Theorem 3.5.The general analysis hence applies and directly leads to the conclusion of the proposition.

(2)For $1/2 < H < 1$, Let us consider $V(x) = x^k$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except X = 0 . Moreover, we have:

$$L\,V \leq kx^{k-1}\left(\left[\alpha + \sigma_2^2 Ht^{2H-1}\right]x - x^3\right)$$
$$- k\left(1-k\right)\sigma_1^2 Ht^{2H-1}x^{3k-2}.$$

Since we assume that $k<1$ and $-kx^{k+2}$ at 0 ,the leading term close to 0 is of order $-k\left(1-k\right)\sigma_1^2 Ht^{2H-1}x^{3k-2}$ which is strictly negative for suciently small x whatever the parameters $\alpha \in \Re$.

(3)For $H = 1/2$. Let us consider $V(x) = x^k$ is $C_2^0(\Re)$ around zero, positive, diverges at zero, we have:

$$LV \le kx^{k-1}\left(\left[\alpha+\frac{\sigma_2^2}{2}\right]x-x^3\right)-k(1-k)\frac{\sigma_1^2}{2}x^{3k-2}$$
$$=k\left(\frac{\sigma_2^2-\sigma_1^2}{2}\right)x^{3k-2}+O\left(x^{3k-2}\right).$$

and hence is negative closed of zero.

Theorem 3.9 For $k>1$. In the case of the supercritical $(\varepsilon=-1)$ stochastic Hopf bifurcation, we have:

(1)When $H=1/2$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $\alpha+\sigma_2^2<0$ and asymptotically stochastically unstable if $\alpha+\sigma_2^2>0$.In that case, there exists a new stochastically stable stationary solution with distribution.

(2)When $0<H<1/2$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $\alpha<0$ and asymptotically stochastically unstable if $\alpha>0$.

(3)When $1/2<H<1$, the null solution of the supercritical Hopf equations is asymptotically stochastically unstable whatever the parameters $\alpha\in\Re$.

Proof. (1) When $H=1/2$. Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$,i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$L\,V \le 2x\left[\left(\alpha+\sigma_2^2\right)x-x^3\right]+\frac{\sigma_1^2}{2}x^{2k}.$$

Since we assume that $k>1$ and $-2x^4$ at 0, the term $L\,V$ is equivalent to $2\left(\alpha+\sigma_2^2\right)x^2$ and hence locally has the sign of $\alpha+\sigma_2^2$. For $\alpha+\sigma_2^2<0$, we can directly apply theorem of [21]. For $\alpha+\sigma_2^2>0$, we use $V(x)=-\log(x)$ again and conclude that 0 is not stable in probability.

(2) When $0<H<0.5$. Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$2x\left[\left(\alpha+\sigma_2^2Ht^{2H-1}\right)x-x^3\right]+2\sigma_1^2Ht^{2H-1}x^{2k}.$$

Since we assume that $k>1$ and $2\sigma^2Ht^{2H-1}x^{2k}$ at 0, the term $L\,V$ is equivalent to $2\alpha x^2$ and hence locally has the sign of α . For $\alpha<0$, we can directly apply theorem $V.4.1$ of [17], x_0 is stable in probability. For $\alpha>0$, we use $V(x)=-\log(x)$ again and conclude that x_0 is not stable in probability.

(3) When $0.5<H<1$. Let us consider $V(x)=x^2$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\Re)$, i.e. it is twice continuously differentiable except at $X=0$. Moreover, we have:

$$2x\left[\left(\alpha+\sigma_2^2Ht^{2H-1}\right)x-x^3\right]+2\sigma_1^2Ht^{2H-1}x^{2k}..$$

Since we assume that $k>1$ and $-2x^4$ at 0, the leading term close to 0 is of order $2\sigma_2^2Ht^{2H-1}x^k$ which is strictly positive for suffciently small x whatever the parameters $\alpha\in\Re$.

4 Example

Example 1. Let us consider the Hopfield neural networks model[23], classically describing the behavior of a cortical column. In that model, the voltage x of a typical neuron in the column is solution of the equation:

$$dx_t=\left(-ax_t+bS\left(x_t\right)\right)dt+g\left(x_t\right)dB_t^H, \qquad (4.1)$$

where $S(x)$ is a smooth sigmoidal function describing the voltage-to-rate transformation, and b is the typical connectivity strength. We choose here $S(x)=\tanh(\gamma x)$ with $\gamma>0$ the sharpness coeffcient of the sigmoid and $g(x_t)=\sigma|x|^k$ with $k\ge\frac{1}{2}$. This equation can be locally reduced to the normal form of the pitchfork bifurcation since the ow is symmetrical,and we obtain using the same transformation as performed above:

$$d\eta_t=\left(\left(-a+b\gamma\right)\eta_t-\eta_t^3+\Psi\left(\eta_t,b,\gamma\right)\right)dt$$
$$+2^{1-k}\sigma|\eta|^k dB_t^H \qquad . \qquad (4.2)$$

When $0.5<H<1$, let us denote by x_0 the stationary solution $\eta_t=0$ a.s. for all t. We have:

- For $k\le 1$, $\eta_t=0$ is asymptotically stable in probability whatever $a,b,\gamma\in\Re$ and $\sigma\ne 0.$
- For $k>1$, $\eta_t=0$ is unstable in probability for $a,b,\gamma\in\Re$ and $\sigma\ne 0$

Example 2. Let us consider a Wilson and Cowan neural network[23] composed of an excitatory and an inhibitory population. The network equations in that case read:

$$\begin{cases} dx_1=\left(-x_1(t)+bS\left(x_1\right)-S\left(x_2\right)\right)dt \\ \qquad +\left(\sigma_1x_1-\sigma_2x_2\right)|x_t|^{k-1}dB_t^H, \\ dx_2=\left(-x_2(t)+bS\left(x_1\right)+S\left(x_2\right)\right)dt \\ \qquad +\left(\sigma_1x_2-\sigma_2x_1\right)|x_t|^{k-1}dB_t^H, \end{cases} \qquad (4.3)$$

where $S(x)=\tanh(\gamma x)$ with $\gamma>0$ and $x_t=\sqrt{x_1^2+x_2^2}$.In the deterministic case, it is straightforward to show that the system undergoes a Hopf bifurcation at $\gamma=1$. Let us now consider the equation satisfied by the modulus $\rho_t=|x_t|$. Using fractional It^o's formula we obtain:

$$\begin{cases} d\rho_t=\left[\left(-1+\gamma\right)+\frac{\sigma_2}{2}\rho_t^{2k-1}+F\left(x_1,x_2\right)\right]dt \\ \qquad +\sigma_1\rho_t^k dB_t^H, \\ d\theta_t=\left[\gamma+\sigma_1\sigma_2\rho^{2k-2}+O\left(\rho_t^2\right)\right]dt+\sigma_2\rho_t^{k-1}dB_t^H, \end{cases}$$
$$(4.4)$$

where

$$F(x_1,x_2) = -\frac{\gamma^3}{3\rho}\left(x_1^4 + x_2^4 + x_1^3x_2 + x_2^3x_1\right) + O\left(\rho^4\right)$$
$$= -\frac{\gamma^3}{3\rho}\left(\left(x_1^2 + x_2^2\right)^2 - 2x_2^2x_1^2\right) + O\left(\rho^4\right)$$
$$\leq -\frac{\gamma^3}{3\rho}\rho^3 + O\left(\rho^4\right).$$

When $0.5 < H < 1$, then the dynamics of the system can hence be analyzed using the above analysis:

- For $k = 1$, the null solution of the supercritical Hopf equations is almost surely exponentially stable if $|\sigma_1| > |\sigma_2|$ and asymptotically stochastically unstable if $|\sigma_1| < |\sigma_2|$.
- For $k < 1$, the null solution of the supercritical Hopf equations is almost surely exponentially stable whatever the parameters $\gamma \in \Re$.
- For $k < 1$, the null solution of the supercritical Hopf equations is asymptotically stochastically unstable whatever the parameters $\gamma \in \Re$.

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