International Journal of New Technology and Research (IJNTR)
ISSN:2454-4116, Volume-1, Issue-4, August 2015 Pages 40-50

Noise and Delay Induced on Dynamics of the Aeroelasticity Model

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***Abstract*. The purpose of this paper is to investigate the stochastic bifurcation and stability problem of the Aeroelasticit-y of two-dimensional supersonic lifting surfaces with delay term. Applying Hopf bifurcation theory, Lyapunov exponent and invariant measure theory, we analyze the D- and P-bifurcation of the stochastic system. The analysis is based on the reduction of the infinite-dimensional problem to one described on a two-dimensional stochastic center manifold.**

Key words: **Stochastic bifurcation; Stochastic stability; Invariant measure; Stochastic aeroelasticity model**

I. Introduction

Because of its evident practical importance, the study of the flutter instability of flight vehicle constitutes an essential prerequisite in their design process. The flutter instability can jeopardize aircraft performance and dramatically affect its survivability. Moreover, the tendency of increasing structural flexibility and maximum operating speed increases the likelihood of the flutter occurrence within the aircraft operational envelope. As a result of the considerable importance of this problem, a great deal of research activity devoted to the aeroelastic active control and flutter suppression of flight vehicles was carried out. In this sense, the reader is referred to a sequence of issues in Refs[1-5], where valuable contributions to this topic have been supplied.

As it clearly appears, within this problem, two principal issues deserve special attention: 1) increase, without weight penalties, of the flutter speed, and 2) possibilities to convert unstable limit cycles into stable ones. While the achievement of 1) can result in the expansion of the flight envelope, the conversions mentioned in 2) would make it possible to operate in close proximity of the flutter boundary without the danger of encountering the catastrophic flutter instability, but in the worst possible scenario, crossing the flutter boundary that features a benign character. In contrast to the catastrophic flutter boundary in which case the amplitude of oscillations increases exponentially, in the case of benign flutter boundary, monotonic increase of the oscillation amplitude occurs in cases 1) and 2) respectively. And, as a result, the failure can occur only by fatigue. It clearly appears that both issues 1) and 2) are related to controlling Hopf bifurcations. In particular, issue 1) implies increase of the stability of an equilibrium and noise of the occurrence of Hopf bifurcations[6-9]whereas issue 2) is related to controlling Hopf bifurcations once a periodic vibration has been initiated[10-15]. Recently, the theory of random dynamical system provides a very powerful mathematical tool for understanding the limiting behavior of stochastic system. It has been applied to engineering, respiratory physiology, chemical plants, mechanical systems, fluid dynamics, secure communications, economics and biological systems[16-21]. Our purpose in this paper is to investigate the stochastic dynamical behavior for the system (2.1) by applying the singular boundary value theory, Lyapunov exponent and the invariant measure theory, the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions are also determined.

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II. Stochastic aeroelastic model

This investigation is based on a geometrical and aerodynamic nonlinear model of a wing section of the high-speed aircraft incorporating an active control capability. As concerns the nonlinear unsteady aerodynamic lift and moment, these are obtained through the integration of the pressure difference and of its moment with respect to the pitching axis, respectively, on the upper and lower surfaces of the airfoil. To this end, the third-order approximation of the piston theory aerodynamics[10-15](PTA), as given by

$$p(x,t)=p_\infty\Big\{1+\kappa(v_z/a_\infty)\gamma+[\kappa(\kappa+1)/4][(v_z/a_\infty)\gamma]^2 \\ +[\kappa(\kappa+1)/12][(v_z/a_\infty)\gamma]^3\Big\}$$

is considered, where κ is polytropic gas coefficient. Here in

$$v_z=-\left(\frac{\partial w}{\partial t}+U_\infty\frac{\partial w}{\partial x}\right)\mathrm{sgn}(z)$$

denotes the downwash velocity normal to the lifting surface $a_\infty^2=\kappa p_\infty/\rho_\infty$ where sgn(z) assumes the value 1 or -1 for $z>0$ and $z<0$, respectively. In addition,

$$w(t)=h(t)+\alpha(t)(x-bx_0)$$

denotes the transversal displacement of the elastic surface; $x_0(\equiv v_{ea})$ is the dimensionless streamwise position of the pitch axis measured from the leading edge; p_∞,ρ_∞, U_∞ and a_∞ are the pressure, the air density, the airflow speed, and the speed of sound of the undisturbed flow, respectively; and $\gamma=M_\infty/\sqrt{M_\infty^2-1}$ is an aerodynamic correction factor that enables one to extend the validity of the PTA to the entire low-supersonic/hypersonic-speed range.

As there also exist many stochastic factors affecting and disturbing the realistic environment considering the change of the twist angle about the pitch axis. We think it is reasonable and necessary to add random terms in the aeroelastic model. In the context of the inclusion of the structural and aerodynamic nonlinearities, of the linear and nonlinear controls and of the associated noise and time delay, in conjunction with the typical cross section with pitch-and plunge degrees of freedom, the dimensionless stochastic aeroelastic equations representing an extension of those in Refs[18,26, 28] are written as

$$\ddot{x}+\chi_{\alpha}\ddot{\alpha}+2\zeta_h\left(\frac{\overline{w}}{v}\right)\dot{x}+\left(\frac{\overline{w}}{v}\right)x=L(t),$$

$$\frac{\chi_{\alpha}}{r_{\alpha}^2}\ddot{x}(t)+\ddot{\alpha}+\frac{1}{V^2}\alpha+\frac{1}{V^2}B\alpha^3=M(t)-\frac{\Psi_1}{V^2}\alpha(t-\tau)$$

$$-\frac{\Psi_2}{V^2}\alpha^3(t-\tau)+\sigma_1(\dot{x}+\dot{\alpha})\xi(t)+\sigma_2\eta(t), \qquad (2.1)$$

where

$$L(t)=-\frac{\gamma}{12\mu M_{\infty}}\{12\alpha+\delta_A M_{\infty}^2(1+\kappa)\gamma^2\alpha^3 +12[\dot{x}+\dot{\alpha}(b-\nu_{ea})/b]\},$$

$$M(t)=-\frac{\gamma}{12\mu M_{\infty}}\{12(b-\nu_{ea})\alpha+\delta_A M_{\infty}^2(b-\nu_{ea}) (1+\kappa)\gamma^3\alpha^2+4[3(b-\nu_{ea})\dot{x}+\dot{\alpha}(4b^2-6b\,\nu_{ea}+3\nu_{ea}^2)/6]\},$$

and $x(t)=h(t)/b(h), \alpha(t)$ is the twist angle about the pitch axis, $\xi(t)$ is the multiplicative random excitation and $\eta(t)$ is the external random excitation directly(namely additive random). $\xi(t)$ and $\eta(t)$ are independent, in possession of zero mean value and standard variance Gauss white noises. i.e. $E[\xi(t)]=E[\eta(t)]=0, E[\xi(t)\xi(t+\tau)] =\delta(\tau), E[\eta(t)\eta(t+\tau)]=\delta(\tau), E[\xi(t)\eta(t+\tau)]=0.$ And (σ_1,α_2) is the intensities of the white noise, and $L(t)$ and $M(t)$ denote the dimensionless aerodynamic lift and moment, respectively. The meaning of the remaining parameters can be found in the nomenclature (see also Refs[1-5,10-15]). In Eq.(2.1), the parameter B identifies the nature of the structural nonlinearity of the system in the sense that, corresponding to $B<0$ or $B>0$, the structural nonlinearities are soft or hard, respectively, whereas for $B=0$ the system is structurally linear. The linear and nonlinear active controls are given in terms of two normalized control gain parameters Ψ_1 and Ψ_2 , respectively.

A mathematical model is generally the first approximation of the considered real system. More realistic models should include some of the stochastic factors affecting and past states of the system, that is, the model should include noise and time delay. The noise and time delay in control can occur either beyond our will or it can be designed as to modify the performance of the system. For this reason, as a necessary prerequisite, a good understanding of its effects on the flutter instability boundary and its character (benign or catastrophic) is required.

To capture the effect of noise $\xi(t)$, $\eta(t)$ and time delay τ , introduced in the related terms Φ_1 and Φ_2 , let $x=x_1$, $\alpha=x_2, \dot{x}=x_3, \dot{\alpha}=x_4, x_{2t}=x_2(t-\tau)$. Then, one can rewrite Eqs.(2.1) as a set four first-order differential equations:

$$\begin{cases}\dot{x}_1=x_3,\\ \dot{x}_2(t)=x_4,\\ \dot{x}_3(t)=a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_2^3+a_6x_{2t}\\ \qquad +a_7x_{2t}^3+(e_1x_3+e_2x_4)\xi(t)+e_3\eta(t),\\ \dot{x}_4(t)=b_1x_1+b_2x_2+b_3x_3+b_4x_4+b_5x_2^3+b_6x_{2t}+b_7x_{2t}^3\\ \qquad +(d_1x_3+d_2x_4)\xi(t)+d_3\eta(t),\end{cases} \qquad (2.2)$$

where $a_6=a_{6c}+\tilde{a}_6$ is the bifurcation parameter, all of the coefficients that are provided in[12]. It is obvious that there exist a unique equilibrium point Q(0, 0, 0, 0).

For convenience in the following analysis, rewrite Eqs.(2.1) in the vector form:

$$\dot{x}(t)=Ax(t)+Bx(t-\tau)+F(x(t),x(t-\tau),\xi(t),\eta(t)),$$

where $x, F\in\Re^4$, A and B are 4×4 matrices. $A, B,$ and F are given by

$$A=\begin{bmatrix}0&0&1&0\\0&0&0&1\\a_1&a_2&a_3&a_4\\b_1&b_2&b_3&b_4\end{bmatrix}, \qquad B=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&a_6&0\\0&b_6&0&0\end{bmatrix},$$

$$F=\begin{bmatrix}0\\0\\a_5x_2^3+a_7x_2^3(t-\tau)+(e_1x_3+e_2x_4)\xi(t)+e_3\eta(t)\\b_5x_2^3+b_7x_2^3(t-\tau)+(d_1x_3+d_2x_4)\xi(t)+d_3\eta(t)\end{bmatrix},$$

respectively.

Hopf bifurcation has been extensively studied using many different methods[23-28] for example, Lyapunov quantity used in the context of the supersonic panel flutter[13-15] where the effects of structural, aerodynamical, and physical nonlinearities have been incorporated. In Refs[11,15] the dynamic behavior of the system without noise and time delay in the control was studied in the vicinity of a Hopf bifurcation critical point. In particular, the effect of the active control on the character of the flutter boundary (where the Jacobian has a purely imaginary pair) is investigated. It is shown that for different flight speeds, stable (unstable) equilibrium and stable (unstable) limit cycles exist.

The effect of the noise and time delay involved in the feedback control will be considered in this paper. Nonlinear systems involving time delay have been studied by many authors[5,12,13]. In the past two decades, there has been rapidly growing interest in bifurcation control[1-5,10-15] There are a wide variety of promising potential applications of bifurcation and chaos control. In general, the aim of bifurcation control is to design a controller such that the bifurcation characteristics of a nonlinear system undergoing bifurcations can be modified to achieve some desirable dynamical behaviors, such as changing a subcritical Hopf bifurcation to supercritical, eliminating chaotic motions, etc. In this context, many applications have been found, for example, in the areas of mechanical systems, fluid dynamics, biological systems, and secure communications. Although effects of delay time on the Aeroelastic model have been extensively investigated [1-5,10-15], there have been no such studies on the effect of multiplicative noise, to the authors' knowledge' . We are interested in the stochastic bifurcation, which is one of the interesting phenomena induced by noise (Refs. [16-21,25-29], related references therein). The main attention is focused on Hopf bifurcation.

As the first step, we analyze the stability of the trivial solution of the linearized system of Eq.(2.3), which is given by

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), x \in \Re. \quad (2.4)$$

The characteristic function can be obtained by substituting the trial solution $x(t) = ce^{\lambda t}$, where c is a constant vector, into the linear part to find

$$\Delta(\lambda, a_6) = \det(\lambda I - A - Be^{-\lambda\tau}) = 0, \quad (2.5)$$

where I denotes the identity matrix. It can be shown[5] that the number of the eigenvalues of the characteristic equation (2.5) with negative real parts, counting multiplicities, can change only when the eigenvalues become pure imaginary pairs as the time delay τ and the components of A and B are varied.

It is seen from Eq.(2.5) that when $5a_1(b_2+b_6) \neq b_1(a_2+a_6)$ none of the roots of $\Delta(\lambda, a_6)$ is zero. Thus, the trivial equilibrium $x=0$ becomes unstable only when Eq.(2.5) has at least a pair of purely imaginary roots $\lambda = \pm\omega i$ (i is the imaginary unit), at which a Hopf bifurcation occurs.

To obtain the explicit analytical expressions for the stability condition of Hopf bifurcation solutions, system (2.1) should be reduced to its center manifold[12,30-32]. While studying the critical infinite dimensional problem on a two-dimensional stochastic center manifold, we express the delay stochastic equation as an abstract stochastic evolution equation on complete probability space. By the centre manifold theorem and Hopf bifurcations[12,30-32], we obtain the equation (2.1) of the stochastic center manifold:

$$\dot{y}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} y + H(y), x(t-\tau) = \Phi(-\tau)y(t), \quad (2.6)$$

$$x(t) = \Phi(0)y(t),$$

where $H(y)$ represents the nonlinear terms contributed from the original system to the stochastic center manifold.

The lowest-order nonlinear terms of the stochastic center manifold, needed to determine the solutions, are

$$H_3(y) = \psi(0)F(\Phi y) = \Phi^{\mathrm{T}}(0) \begin{bmatrix} 0 \\ 0 \\ a_5(\Phi(0)y)_2^3 + a_7(\Phi(-\tau)y)_2^3 + (e_1\Phi(0)y + e_2\Phi(0) \\ y)\xi(t) + e_3\eta(t) \\ b_5(\Phi(0)y)_2^3 + b_7(\Phi(\tau)y)_2^3 + (d_1\Phi(0)y + d_2\Phi(0) \\ y)\xi(t) + d_3\eta(t) \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}y_1^3 + c_{12}y_1^2y_2 + c_{13}y_1y_2^2 + c_{14}y_2^3 + (\hat{e}_1 y_1 + \hat{e}_2 y_2) \\ \xi(t) + \hat{e}_3\,\eta(t) \\ c_{21}y_1^3 + c_{22}y_1^2y_2 + c_{23}y_1y_2^2 + c_{24}y_2^3 + (\hat{d}_1 y_1 + \hat{d}_2 y_2) \\ \xi(t) + \hat{d}_3\,\eta(t) \end{bmatrix},$$

where

$$\Phi(\theta) = \begin{bmatrix} \cos\omega\theta & \sin\omega\theta \\ \frac{L_1\cos\omega\theta + \omega L_2\sin\omega\theta}{L_0} & \frac{L_1\sin\omega\theta - \omega L_2\cos\omega\theta}{L_0} \\ -\omega\sin\theta & \omega\cos\omega\theta \\ \frac{\omega(\omega L_2\cos\omega\theta - L_1\sin\omega\theta)}{L_0} & \frac{\omega(\omega L_2\cos\omega\theta + L_1\sin\omega\theta)}{L_0} \end{bmatrix},$$

$$\Psi(\theta) = \begin{bmatrix} \frac{L_3\cos\omega\theta + L_4\sin\omega\theta}{M} & \frac{L_3\sin\omega\theta - L_4\cos\omega\theta}{M} \\ \frac{L_5\cos\omega\theta + L_6\sin\omega\theta}{M} & \frac{L_5\sin\omega\theta - L_6\cos\omega\theta}{M} \\ \frac{L_7\cos\omega\theta + L_8\sin\omega\theta}{M} & \frac{L_7\sin\omega\theta - L_8\cos\omega\theta}{M} \\ N_1\cos\omega\theta - N_2\sin\omega\theta & N_1\sin\omega\theta + N_2\cos\omega\theta \end{bmatrix},$$

where the explicit expressions of $L_i (i = 1,2,..,8)$ and N are also provided in[12], and N_1 and N_2 can be obtained from the relation $\langle \Psi, \Phi \rangle = I$, expressed in terms of ω, τ and the coefficients a_i, b_i, d_i, and e_i in Eqs.(2.1). The lengthy expressions of $c_{ij}, \hat{e}_i$ and d_i are omitted here. Therefore we obtain the equation (2.1) of the stochastic center manifold:

$$\begin{cases} \dot{y}_1(t) = \omega y_2 + c_{10}y_1 + c_{100}y_2 + c_{11}y_1^3 + c_{12}y_1^2y_2 + \\ \qquad c_{13}y_1y_2^2 + c_{14}y_2^3 + (\hat{e}_1y_1 + \hat{e}_2y_2)\xi(t) + \hat{e}_3\eta(t), \\ \dot{y}_2(t) = -\omega y_1 + c_{20}y_1 + c_{200}y_2 + c_{21}y_1^3 + c_{22}y_1^2y_2 + \\ \qquad c_{23}y_1y_2^2 + c_{24}y_2^3 + (\hat{d}_1y_1 + \hat{d}_2y_2)\xi(t) + \hat{d}_3\eta(t). \end{cases}$$

We set the coordinate transformation $y_1 = r\cos\theta, y_2 = r\sin\theta$, and by substituting the variable in (2.7), we obtain

$$\begin{cases} \dot r(t) = rc_{10}\cos^2\theta + r(c_{100}+c_{20})\sin\theta\cos\theta + rc_{200} \\ \quad \sin^2\theta r^3[c_{11}\cos^4\theta + (c_{12}+c_{21})\cos^3\theta\sin\theta \\ \quad +(c_{13}+c_{22})\cos^2\theta\sin^2\theta + (c_{14}+c_{23})\cos\theta \\ \quad \sin^3\theta + c_{24}\sin^4\theta] + r[\hat e_1\cos^2\theta + (\hat e_2+\hat d_1) \\ \quad \cos\theta\sin\theta + \hat d_2\sin^2\theta]\xi(t) + [\hat e_3\cos\theta + \hat d_3 \\ \quad \sin\theta]\eta(t), \\ \dot\theta(t) = -\omega + c_{20}\cos^2\theta + (c_{200}-c_{10})\sin\theta\cos\theta \\ \quad -c_{100}\sin^2\theta + r^2[c_{21}\cos^4\theta + (c_{21}-c_{12}) \\ \quad \cos^3\theta\sin\theta + (c_{23}+c_{12})\cos^2\theta\sin^2\theta + \\ \quad (c_{24}+c_{213})\cos\theta\sin^3\theta + c_{14}\sin^4\theta] + [\hat d_1 \\ \quad \cos^2\theta + (\hat d_2-\hat e_1)\cos\theta\sin\theta - \hat e_2\sin^2\theta] \\ \quad \xi(t) + \frac{1}{r}[\hat d_3\cos\theta + \hat e_3\sin\theta]\eta(t). \end{cases} \tag{2.8}$$

It is difficult to calculate the exact solution for system (2.8) today. According to the Khasminskii limit theorem, when the intensities of the white noises $(e_i, d_i)(i=1,2,3)$ is small en-enough, the response process $\{r(t),\theta(t)\}$ weakly converged to the two-dimensional Markov diffusion process [26-29]. Through the stochastic averaging method, stochastic differential equations (2.9) are obtained

$$\begin{cases} dr = m_r dt + \sigma_{11}dW_r + \sigma_{12}dW_\theta, \\ d\theta = m_\theta dt + \sigma_{21}dW_r + \sigma_{22}dW_\theta, \end{cases} \tag{2.9}$$

where $W_r(t)$ and W_θ are the independent and standard Wiener processes. As for the twodimensional diffusion process, it is necessary to calculate its two-dimensional transition probability density. There is no general and right method for the calculation. As for the concrete system, we could finish the calculation with some special techniques.
Set the parameters as follows:

$$\mu_1 = \frac{1}{2}(b_{10}+b_{200}),\quad \mu_{10} = -\omega + \frac{1}{2}(b_{20}+b_{100}),$$
$$\mu_2 = 5\hat e_1^2 + 5\hat d_2^2 + 3\hat d_1^2 + 3\hat e_2^2 + 6\hat e_2\hat d_1 - 2\hat e_1\hat d_2,$$
$$\mu_3 = \frac{1}{2}(\hat e_3^2 + \hat d_3^2),\quad \mu_4 = 3\hat e_1^2 + 3\hat d_2^2 + \hat e_2^2 + \hat d_1^2 + 2\hat e_2\hat d_1 + 2\hat e_1\hat d_2,$$
$$\mu_5 = \frac{1}{4}(\hat e_1+\hat d_2)(\hat d_1-\hat e_2),\quad \mu_6 = \hat e_1^2 + \hat d_2^2 + 3\hat e_2^2 + 3\hat d_1^2 - 2\hat e_2\hat d_1 - 2\hat e_1\hat d_2,$$
$$\mu_7 = 3c_{11} + 3c_{21} + c_{13} + c_{22},\ \mu_8 = 3c_{21} - 3c_{14} + c_{23} - c_{12}.$$

Under the condition $\sigma_{12}^2 = \sigma_{21}^2 \neq 0$, we rewrote system (2. 9) as follows

$$\begin{cases} dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_3}{r} + \frac{\mu_7}{8}r^3\right]dt + \left(\mu_3 + \frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r + (r\mu_5)^{\frac{1}{2}}dW_\theta, \\ d\theta = \left(\mu_{10} + \frac{\mu_8}{8}r^2\right)dt + (r\mu_5)^{\frac{1}{2}}dW_r + \left(\frac{\mu_3}{r^2} + \frac{\mu_6}{8}\right)^{\frac{1}{2}} dW_\theta. \end{cases} \tag{2.10}$$

From the diffusion matrix, we can find that the averages amplified $r(t)$ is a one-dimensional Markov diffusion process when $\sigma_{12}^2 = \sigma_{21}^2 = 0$, *i.e.* $\hat e_1 + \hat d_2 = 0$, or $\hat d_1 - \hat e_2 = 0$ .Thus we have the equation as following

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_3}{r} + \frac{\mu_7}{8}r^3\right]dt + \left(\mu_3 + \frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{2.11}$$

This is an efficient method to obtain the critical point of stochastic bifurcation through analyzing the change of stability of the averaging amplitude $r(t)$ in the meaning of probability.

III. STOCHASTIC D-BIFURCATION

In the section, We will see how the introduction of randomness change the stochastic behavior significantly from both the dynamical and phenomenological points of view[26-29].

Theorem 3.1 (D-Bifurcation) When $\mu_3 = 0, \mu_7 = 0.$ Then the delayed stochastic system (2.2) undergoes a D-bifurcation, at the parameter value $16\mu_1 + 2\mu_2 = \mu_4$. But the stochastic system (2.2) does not undergo P-bifurcation.

Proof. When $\mu_3 = 0, \mu_7 = 0.$ Then system (2.2) becomes

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}} dW_r. \tag{3.1}$$

When $\mu_4 = 0$, equation (3.1) is a determinate system, and there is no bifurcation phenomenon.. Here we discuss the situation $\mu_4 \neq 0$, let

$$m(r) = \left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)r,\qquad \sigma(r) = \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} r.$$

The continuous random dynamic system generated by (3.1) is

$$\phi(t)x = x + \int_0^t m(\phi(s)x)ds + \int_0^t \sigma(\phi(s)x)\circ dW_r.$$

where $\circ dW_r$ is the differential in the meaning of Stratanovich, it is the unique strong solution of (3.1) with initial value x. And $m=0, \sigma(0)=0$, so 0 is a fixed point of φ. Since $m(r)$ is bounded and for any $r \neq 0$, it satisfies the ellipticity condition: $\sigma(r) \neq 0$; it is assured that there is at most one stationary probability density. According to the $It\hat{o}$ equation of amplitude $r(t)$, we obtain its FPK equation corresponding to (3.1) as follows

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}\left\{\left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]p\right\} + \frac{\partial^2}{\partial r^2}\left\{\left[\frac{\mu_4}{8}r^2\right]p\right\}. \tag{3.2}$$

Let $\frac{\partial p}{\partial t} = 0$, then we obtain the solution of system (3.2)

$$p(t) = c\left|\sigma^{-1}(t)\right|\exp\left(\int_0^t \frac{2m(\mu)}{\sigma^2(\mu)}d\mu\right). \tag{3.3}$$

The above dynamical system (3.2) has two kinds of equilibrium state: fixed point and nonstationary Motion. The invariant measure of the former is δ_0 and it's probability density is δ_x. The invariant measure of the latter is ν and it's probability density is (3.3). In the following, we calculate the lyapunov exponent of the two invariant measures.

Using the solution of linear $It\hat{o}$ stochastic differential equation, we obtain the solution of system (3.1).

$$r(t) = r(0)$$
$$\exp\left(\int_0^t\left[m'(0) + \frac{\sigma(0)\sigma'(0)}{2}\right]ds + \int_0^t \sigma'(0)dW_r.\right) \tag{3.4}$$

The Lyapunov exponent with regard to μ of dynamic system φ is defined as:

$$\lambda_\varphi(\mu) = \lim_{t\to+\infty}\frac{1}{t}\ln\|r(t)\|, \tag{3.5}$$

Substituting (3.4) into (3.5), note that $\sigma(0) = 0, \sigma'(0) = 0$, we obtain the Lyapunov exponent of the fixed point:

$$\lambda_\varphi(\delta_0)$$
$$= \lim_{t\to+\infty}\frac{1}{t}\left(\ln\|r(0)\| + m'(0)\int_0^t ds + \sigma'(0)\int_0^t dW_r(s)\right)$$
$$= m'(0) + \sigma'(0)\lim_{t\to+\infty}\frac{W_r(t)}{t}$$
$$= m'(0)$$
$$= \mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}. \tag{3.6}$$

For the invariant measure which regard (3.4) as its density, we obtain the Lyapunov exponent:

$$\lambda_\phi(\nu) = \lim_{t\to+\infty}\frac{1}{t}\int_0^t\left[m'(r) + \sigma(r)\sigma'(r)\right]ds$$
$$= \int_R\left[m'(r) + \frac{\sigma(r)\sigma'(r)}{2}\right]p(r)dr$$
$$= -2\int_R\left[\frac{m(r)}{\sigma(r)}\right]^2 p(r)dr$$
$$= -32\sqrt{2}\mu_4^{\frac{3}{2}}m(r)^2\exp\left[\frac{16}{\mu_4}m(r)\right]$$
$$= -32\sqrt{2}\mu_4^{\frac{3}{2}}\left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)^2$$
$$\exp\left[\frac{16}{\mu_4}\left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)\right]. \tag{3.7}$$

Let $\alpha = \mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}$. We can obtain that the invariant measure of the fixed point is stable when $\alpha < 0$, but the invariant measure of the non-stationary motion is stable when $\alpha > 0$, so $\alpha = \alpha_D = 0$ is a point of D-bifurcation.

Simplify Eq.(3.3), we can obtain

$$p_{st}(r) = cr^{\frac{2(8\mu_1+\mu_2-\mu_4)}{\mu_4}}, \tag{3.8}$$

where c is a normalization constant, thus we have

$$p_{st}(r) = o(r^\nu) \qquad r \to 0, \tag{3.9}$$

where $\nu = \frac{2(8\mu_1 + \mu_2 - \mu_4)}{\mu_4}$. Obviously when $\nu < -1$, that is $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} < 0$, $p_{st}(r)$ is a δ function. when $-1 < \nu < 0$, that is $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} > 0, r = 0$ is a maximum point of $p_{st}(r)$ in the state space, thus the system undergoes D-bifurcation when $\nu = -1$, that is $\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16} = 0$, is the critical condition of D-bifurcation at the equilibrium point. When $\nu > 0$, there is no point that makes $p_{st}(r)$ have maximum value, thus the system does not undergo P-bifurcation.

Theorem 3.2(Stochastic Pitchfork bifurcation) When $\mu_3 = 0, \mu_7 < 0$. Then the system (2.2) undergoes stochastic pitchfork bifurcation.

Proof. When $\mu_3 = 0, \mu_7 \neq 0$. then Eq(2.11) can rewrite as follows

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_7}{8}r^3\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}dW_r. \tag{3.10}$$

Let $\phi = \sqrt{\frac{-\mu_7}{8}}r, \mu_7 < 0$,then the system (3.10) becomes

$$d\varphi = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)\varphi - \varphi^3\right]dt + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}\varphi \circ dW_t \tag{3.11}$$

which has the solution

$$\phi \to \psi_{\mu_1}(t,\omega)\phi = \frac{\phi \exp\left(\left(\mu_1+\frac{\mu_2}{8}\right)t+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_t(\omega)\right)}{\left(1+2\phi^2\int_0^t \exp\left(2\left(\left(\mu_1+\frac{\mu_2}{8}\right)t+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)\right)ds\right)^{\frac{1}{2}}} \quad (3.12)$$

We now determine the domain $D_{\mu_1}(t,\omega)$, where $D_{\mu_1}(t,\omega) := \{\varphi \in \Re : (t,\omega,\varphi) \in D\} (D = \Re \times \Omega \times X)$ is the (in general possibly empty) set of initial values $\phi \in \Re$ for which the trajectories still exist at time t and the range $R_{\mu_1}(t,\omega)$ of $\psi_{\mu_1}(t,\omega) : D_{\mu_1}(t,\omega) \to R_{\mu_1}(t,\omega)$.

We have

$$D_{\mu_1}(t,\omega) = \begin{cases} \Re, & t \geq 0, \\ \left(-d_{\mu_1}(t,\omega), d_{\mu_1}(t,\omega)\right), & t < 0, \end{cases} \quad (3.13)$$

where

$$d_{\mu_1}(t,\omega) = \frac{1}{\left(2\left|\int_0^t \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t+2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)ds\right|\right)^{\frac{1}{2}}} > 0,$$

and

$$R_{\mu_1}(t,\omega) = D_{\mu_1}(-t,\vartheta(t)\omega) = \begin{cases} \left(-r_{\mu_1}(t,\omega), r_{\mu_1}(t,\omega)\right), t > 0, \\ \Re, \quad t \leq 0, \end{cases} \quad (3.14)$$

where

$$r_{\mu_1}(t,\omega) = d_{\mu_1}(-t,\vartheta(t)\omega) = \frac{\exp\left(\left(\mu_1+\frac{\mu_2}{8}\right)t+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_t(\omega)\right)}{\left(2\left|\int_0^t \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t+2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)ds\right|\right)^{\frac{1}{2}}} > 0.$$

We can now determine

$$E_{\mu_1}(\omega) := \cap_{t\in\Re} D_{\mu_1}(t,\omega)$$

and obtain

$$E_{\mu_1}(\omega) = \begin{cases} \left(-d_{\mu_1}^-(t,\omega), d_{\mu_1}^-(t,\omega)\right), & \mu_1+\frac{\mu_2}{8} > 0, \\ \{0\}, & \mu_1+\frac{\mu_2}{8} \leq 0, \end{cases}$$

where

$$0 < d_{\mu_1}^{\pm}(t,\omega) = \frac{1}{\left(2\left|\int_0^{\pm\infty} \exp\left(2\left(\left(\mu_1+\frac{\mu_2}{8}\right)t+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_s(\omega)\right)\right)ds\right|\right)^{\frac{1}{2}}} < \infty$$

The ergodic invariant measures of system (3.10) are

(i) For $\mu_1+\frac{\mu_2}{8} \leq 0$, the only invariant measures is $\mu_\omega^{\mu_1} = \delta_0$.

(ii) For $\mu_1+\frac{\mu_2}{8} > 0$ we have the three invariant forward Markov measures $\mu_\omega^{\mu_1} = \delta_0$ and $\upsilon_{\pm,\omega}^{\mu_1} = \delta_{\pm k_{\mu 1}(\omega)}$, where

$$k_{\mu_1}(\omega) := \left(2\int_{-\infty}^0 \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t+2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}}W_t(\omega)\right)ds\right)^{-\frac{1}{2}}.$$

We have $Ek_{\mu_1}^2(\omega) = \alpha$. Solving the forward Fokker Planck equation

$$L^* p_{\mu_1} = -\left(\left(\left(\mu_1+\frac{\mu_2}{8}\right)\phi+\frac{\mu_4}{16}\phi-\phi^3\right)P_{\mu_1}(\phi)\right)' + \frac{\mu_4}{16}\left(\phi^2 P_{\upsilon 1}(\phi)\right)'' = 0$$

yields

(i) $p_{\mu_1} = \delta_0$ for all $\mu_1+\frac{\mu_2}{8}$,

(ii) for $p_{\mu_1} > 0$

$$q_{\mu_1}^+(\phi) = \begin{cases} N_{\mu_1}\phi^{\frac{2(\mu_1+\frac{\mu_2}{8})}{\mu_4}-1}\exp\left(\frac{\phi^2}{\mu_4}\right), & \phi > 0, \\ 0, & \phi \leq 0, \end{cases}$$

and $q_{\mu_1}^+(\phi) = q_{\mu_1}^+(-\phi)$, where $N_{\mu_1}^- = \Gamma\left(\frac{\upsilon_1}{\mu_4}\right)\left(\frac{\mu_4}{8}\right)^{\left(\mu_1+\frac{\mu_2}{8}\right)}\mu_4$.

Naturally the invariant measures $\nu^{\mu_1}_{\pm,\omega} = \delta_{\pm k_{\mu_1}(\omega)}$. are those corresponding to the stationary measures $q^{+}_{\mu 1}$. Hence all invariant measures are Markov measures.

We determine all invariant measures(necessarily Dirac measure) of local RDS χ generated by the SDE

$$d\phi = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)\phi - \phi^3\right]dt + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} \phi \circ dW. \quad (3.16)$$

on the state space $\Re$, $\mu_1 + \frac{\mu_2}{8} \in \Re$ and $\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} \geq 0$. We now calculate the Lyapunov exponent for each of these measure.

The linearized RDS $\chi_t = D\Upsilon(t,\omega,\phi)\chi$ satisfies the linearized SDE

$$d\chi_t = \left[\left(\mu_1 + \frac{\mu_2}{8}\right) - 3\left(\Upsilon(t,\omega,\phi)\right)^2 \chi_t\right]dt$$

$$+\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} \chi_t \circ dW.$$

Hence

$$D\Upsilon(t,\omega,\phi)\chi = \chi \exp$$

$$\left(\left(\mu_1 + \frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_t(\omega) - 3\int_0^t \left(\Upsilon(s,\omega,\phi)\right)^2 ds\right).$$

Thus, if $\nu_\omega = \delta_{\varphi_0(\omega)}$ is a $\Upsilon-$ invariant measure, its Lyapunov exponent is

$$\lambda(\mu) = \lim_{t\to\infty}\frac{1}{t}\log\left\|D\Upsilon(t,\omega,\phi)\right\|$$

$$= \mu_1 + \frac{\mu_2}{8} - 3\lim_{t\to\infty}\int_0^t \left(\Upsilon(s,\omega,\phi)\right)^2 ds$$

$$= \mu_1 + \frac{\mu_2}{8} - 3\mathrm{E}\,\phi_0^2,$$

provided the IC $\phi_0^2 \in L^1(\mathrm{P})$ is satisfied.

(i) For $\mu_1 + \frac{\mu_2}{8} \in \Re$ the IC for $\nu^{\mu_1}_{,\omega} = \delta_0$ is trivially satisfied and we obtain

$$\lambda\left(\nu_1^{\mu_1}\right) = \mu_1 + \frac{\mu_2}{8}.$$

So $\nu_1^{\mu_1}$ is stable for $\mu_1 + \frac{\mu_2}{8} < 0$ and unstable for $\mu_1 + \frac{\mu_2}{8} > 0.$

(ii) For $\mu_1 + \frac{\mu_2}{8} > 0, \nu^{\mu_1}_{2,\omega} = \delta_{d^{\mu_1}_{\omega}}$ is $\mathrm{F}^{0}_{-\infty}$ measurable, hence the density p^{μ_1} of $\mathrm{p}^{\mu_1} = \mathrm{E}\nu_2^{\mu_1}$ satisfies the Fokker-Planck equation

$$L^*_{\nu_1} = -\left(\left(\left(\mu_1 + \frac{\mu_2}{8}\right)\phi + \frac{\mu_4}{16}\phi - \phi^3\right)p_{\mu_1}(\phi)\right)'$$

$$+\frac{\mu_4}{16}\left(\phi^2 p_{\mu_1}(\phi)\right)'' = 0,$$

which has the unique probability density solution

$$P^{\mu_1}(\phi) = N_{\mu_1}\phi^{\frac{2\left(\mu_1 + \frac{\mu_2}{8}\right)}{\mu_4} - 1} \exp\left(\frac{8\phi^2}{\mu_4}\right), \quad \phi > 0.$$

Since

$$\mathrm{E}_{\nu_2^{\mu_q}}\varphi^2 = \mathrm{E}\left(d_-^{\mu_1}\right)^2 = \int_0^\infty \varphi^2 p^{\mu_1}(\varphi)d\varphi < \infty,$$

the IC is satisfied. The calculation of the Lyapunov exponent is accomplished by observing that

$$d_-^{\mu_1}(\upsilon_t\omega)^2 = \frac{\exp\left(2\left(\mu_1 + \frac{\mu_2}{8}\right)t + 2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_t(\omega)\right)}{2\int_{-\infty}^t \exp\left(2\left(\mu_1 + \frac{\mu_2}{8}\right)s + 2\left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_s(\omega)\right)ds}$$

$$= \frac{\Psi'(t)}{2\Psi},$$

where

$$\Psi(t) = \int_{-\infty}^t \exp\left(\left(\mu_1 + \frac{\mu_2}{8}\right)s + \left(\frac{\mu_4}{8}\right)^{\frac{1}{2}} W_s(\omega)\right)ds.$$

Hence by the ergodic thoerem

$$\mathrm{E}\left(d_-^{\mu_1}\right)^2 = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\log\Psi(t) = \mu_1 + \frac{\mu_2}{8},$$

finally

$$\lambda\left(\nu_2^{\mu_1}\right) = -2\left(\mu_1 + \frac{\mu_2}{8}\right) < 0.$$

(iii) For $\mu_1 + \frac{\mu_2}{8} > 0, \nu^{\mu_1}_{2,\omega} = \delta_{d^{\mu_1}_{\omega}}$ is $F^{0}_{-\infty}$ measurable. Since

$$\mathrm{L}\left(d_+^{\mu_1}\right) = \mathrm{L}\left(d_-^{\mu_1}\right), \mathrm{E}\left(-d_-^{\mu_1}\right)^2 = \mathrm{E}\left(d_-^{\mu_1}\right)^2 = \mu_1 + \frac{\mu_2}{8}$$

thus

$$\lambda\left(\nu_2^{\mu_1}\right) = -2\left(\mu_1 + \frac{\mu_2}{8}\right) < 0.$$

The two families of densities $\left(q^{+}_{\mu_1}\right)_{\mu_1} > 0$ clearly undergo a P-bifurcation at the parameter value $\mu_{1P} = \frac{\mu_4}{8}$ Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $\mu_D = 0$ and a P-bifurcation of $\mu_P = \dfrac{\left(\mu_1 + \dfrac{\mu_2}{8}\right)^2}{2}$.

IV. P-BIFURCATION

In the following, we consider the steady-state probability density $p_{st}(r)$ of the linear $It\hat{o}$ stochastic differential equation. Calculating extreme values of the invariant measure is one of the most popular efficient methods in studying the bifurcation of a nonlinear dynamical system. The invariant measure is an important characteristic value of stochastic bifurcation.

4.1 Case I: $\mu_3 \neq 0, \mu_7 = 0$

When $\mu_3 \neq 0, \mu_7 = 0$. According to the $It\hat{o}$ equation of amplitude $r(t)$, we obtain its FPK equation as follows

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}\left\{\left[(\mu_1 + \frac{\mu_2}{8})r + \frac{\mu_3}{r}\right]p\right\} + \frac{\partial^2}{\partial r^2}\left\{\left[\left(\mu_3 + \frac{\mu_4}{8}r^2\right)\right]p\right\} \tag{4.1}$$

with the initial value Condition $\mu_7 = 0, p(r,t \mid r_0,t_0) \rightarrow$, $\delta(r - r_0), t \rightarrow t_0$, where $p(r,t \mid r_0,t_0)$ is the transition probability density of diffusion process $r(t)$. The invariant measure of $r(t)$ is the steady-state probability density $p_{st}(r)$ which is the solution of the degenerate system as following

$$0 = -\frac{\partial}{\partial r}\left\{\left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_3}{r}\right]p\right\} + \frac{\partial^2}{\partial r^2}\left\{\left[\left(\mu_3 + \frac{\mu_4}{8}r^2\right)\right]p\right\} \tag{4.2}$$

Through calculation, we can obtain

$$p_{st}(r) = 4\sqrt{\frac{2}{\pi}}2^{-3\nu}\mu_3^{2-\nu}\left(\frac{\mu_4}{\mu_3}\right)^{\frac{3}{2}}\Gamma(2-\nu)\left(\Gamma\left(\frac{1}{2}-\nu\right)\right)^{-1} r^2\left(\mu_4 r^2 + 8\mu_3\right)^{\nu-2} \tag{4.3}$$

where $\nu = (8\mu_1 + \mu_2)\mu_4^{-1}, \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$

According to Namachivaya's theory[27], the extreme value of an invariant measure contains the most important essence of the nonlinear stochastic system. In other words, the invariant measure can uncover the characteristic information of the steady state. When the intensity of noise tends to zero, the extreme values of $p_{st}(r)$ approximately show the behavior of the deterministic system. If the process $r(t)$ is ergodic then $p_{st}(r)$ can be regarded as the time measurement for staying in the neighborhood of $a(t)$ according to Oseled-
ec ergodic theorem.

From the analysis above, we know that the parameters $\mu_3 > 0$, $\mu_2 > \mu_4 > 0$. If $p_{st}(r)$ has a maximum value at r^*, the sample trajectory will stay for a longer time in the neighborhood of r^*, i.e. r^* is stable in the meaning of probability (with a bigger probability). If $p_{st}(r)$ has a minimum value (zero), it is just the opposite.

We now calculate the most possible amplitude r^* of system (2.9)., i.e. $p_{st}(r)$ has a maximum value at r^*. So we have

$$\left.\frac{dp_{st}(r)}{dr}\right|_{r=r^*} = 0, \qquad \left.\frac{d^2 p_{st}(r)}{dr^2}\right|_{r=r^*} < 0$$

and the solution $r = 0$ or

$$r^* = \tilde{r} = \sqrt{\frac{-8\mu_3}{8\mu_1 + \mu_2 - \mu_4}}\left(as\ \frac{8\mu_1 + \mu_2}{\mu_4} < \frac{1}{2}\right).$$

Further, we have

$$\left.\frac{d^2 p_{st}(r)}{dr^2}\right|_{r=0} = 2^{6+3(8\mu_1+\mu_2-\mu_4)\mu^{-1}}\mu_3^{2+(8\mu_1+\mu_2-\mu_4)\mu^{-1}} > 0$$

$$\left.\frac{d^2 p_{st}(r)}{dr^2}\right|_{r=\tilde{r}} = \frac{(8\mu_1 + \mu_2 - \mu_4)^3\left(8\mu_3 - \dfrac{8\mu_3\mu_4}{8\mu_1 + \mu_2 - \mu_4}\right)^{\frac{8\mu_1+\mu_2}{\mu_4}}}{-16(8\mu_1 + \mu_2 - \mu_4)^3} < 0.$$

Thus what we need is $r^* = \tilde{r}$. In the meantime, $p_{st}(r)$ is 0 (minimum) at $r = 0$, . This means that the system subjected to random excitations is almost unsteady at the equilibrium point $(r = 0)$ in the meaning of probability. The conclusion is to go all the way with what has been obtained by the singular boundary theory. The original nonlinear stochastic system has a stochastic Hopf bifurcation at $r = \tilde{r}$.

$$x_1^2 + x_2^2 = \frac{-8\mu_3}{8\mu_1 + \mu_2 - \mu_4}, \qquad (i.e.\quad r = \tilde{r}).$$

We now choose some values of the parameters in the equations, draw the graphics of $p_{st}(r)$. The curves in the graph belonging to the cond1,2,3,4 in turn are shown in Fig. 1a. It is worth putting forward that calculating the Hopf bifurcation with the parameters in the original system is necessary. If we now have values of the original parameters in system (2.1), that $b=1.5, \mu=1, \bar{\omega}=0.8, r_\alpha=0.5625, \chi_\alpha=0.22, \zeta_h=\zeta_\alpha=0.1, \gamma_1=1.3, \kappa=1.5, \delta_A=0.1, B=1, x_0=0.1, \sigma_1=0.1, \sigma_2=0.1$. After further calculations we obtain $\mu_1=-0.7625$, $\mu_2=4.26167$, $\mu_3=0.324941$, $\mu_4=2.56999$,

$$\nu=\frac{8\mu_1+\mu_2}{\mu_4}=-0.715307<\frac{1}{2},$$

$$p(r)=\frac{25.3905r^2}{(2.59953+2.56999r^2)^{2.71531}}.$$

What is more is that $\tilde{r}=0.767911$ where $p_{st}(r)$ has the maximum value(see Fig.1b).

4.2 Case II: $\mu_3=0, \mu_7\neq 0.$

When $\mu_3=0, \mu_7\neq 0.$ then Eq(2.9) can rewrite as following

$$dr=\left[\left(\mu_1+\frac{\mu_2}{8}\right)r+\frac{\mu_7}{8}r^3\right]dt+\left(\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}dw_r \quad (4.4)$$

According to the $It\hat{o}$ equation of amplitude $r(t)$, we obtain its FPK equation form (4.4) as following

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial r}\left\{\left[\left(\mu_1+\frac{\mu_2}{8}\right)r-\frac{\mu_4}{2\mu_7}r-r^3\right]p(r)\right\}-\frac{\mu_4}{2\mu_7}\frac{\partial^2}{\partial r^2}\left(r^2p(r)\right) \quad (4.5)$$

with the initial value condition $\mu_7=0, p(r,t|r_0,t_0)\to\delta(r-r_0), t\to t_0$, where $p(r,t|r_0,t_0)$ is the transition probability density of diffusion process $r(t)$. The invariant measure of $r(t)$ is the steady- state probability density $p_{st}(r)$ which is the solution of the degenerate system as following

$$0=-\frac{\partial}{\partial r}\left\{\left[\left(\mu_1+\frac{\mu_2}{8}\right)r-\frac{\mu_4}{2\mu_7}r-r^3\right]p(r)\right\}-\frac{\mu_4}{2\mu_7}\frac{\partial^2}{\partial r^2}\left(r^2p(r)\right). \quad (4.6)$$

Through calculation, we can obtain

$$p_{st}(r)=\frac{\exp\left(\frac{r^2\mu_7}{\mu_4}\right)r^{-1-\frac{2\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{\mu_4}}}{\Gamma\left[-\frac{2\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{\mu_4}\right]\left(-\frac{\mu_7}{\mu_4}\right)^{\frac{\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{\mu_4}}} \quad (4.7)$$

According to Namachivaya's theory[27], the extreme value of an invariant measure contains the most important essence of the nonlinear stochastic system. In other words, the invariant measure can uncover the characteristic information of the steady state. When the intension of the noise tends to zero, the extreme values of $p_{st}(r)$ approximate to show the behavior of the deterministic system. If the process $r(t)$ is ergodic then $p_{st}(r)$ can be regarded as the time mea- surement for staying in the neighborhood of $a(t)$ according to Oseledec ergodic theorem.

From the analysis above, if $p_{st}(r)$ has a maximum value at r^*, the sample trajectory will stay for a longer time in the neighborhood of r^*, i.e. r^* is stable in the meaning of probability (with a bigger probability). If $p_{st}(r)$ has a minimum value (zero), it is just the opposite.

We now calculate the most possible amplitude r^* of system (4.4)., i.e. $p_{st}(r)$ has a maximum value at r^*. So we have

$$\left.\frac{dp_{st}(r)}{dr}\right|_{r=r^*}=0, \qquad \left.\frac{d^2p_{st}(r)}{dr^2}\right|_{r=r^*}<0$$

and the solution $r=\tilde{r}=\sqrt{\frac{\nu_1\nu_2+4\nu_3}{8\nu_2}}$. The probabilities and the positions of the Hopf bifurcation occurrence with different parameter are listed, and the corresponding results can be seen in Fig.2 as well.

Since

$$\left.\frac{d^2p_{st}(r)}{dr^2}\right|_{r=\tilde{r}}=2^{4+\frac{3(\mu_1+\frac{\mu_2}{8})\mu_2}{\mu_3}}\exp\left(\frac{1}{8}\left(4+\frac{\mu_7}{\mu_3}\right)\right)\mu_7\left(-\frac{\mu_7}{\mu_4}\right)^{-\frac{\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{\nu_4}}\left(\sqrt{\frac{8\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7+4\mu_4}{\mu_7}}\right)^{-\frac{2\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{4\mu_4}}\Bigg/\Gamma\left[-\frac{\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{\mu_4}\right]\left(-\frac{\mu_7}{\mu_4}\right)^{\frac{\left(\mu_1+\frac{\mu_2}{8}\right)\mu_7}{\mu_4}}<0.$$

Thus what we need is $r^*=\tilde{r}$. The conclusion is to go all the way with what has been obtained by the singular boundary theory. The original nonlinear stochastic system has a stochastic Hopf bifurcation at $r = \tilde{r}$.

$$x_1^{\,2} + x_2^{\,2} = \frac{8\left(\mu_1 + \frac{\mu_2}{8}\right)\mu_7 + 4\mu_4}{8\mu_7}$$

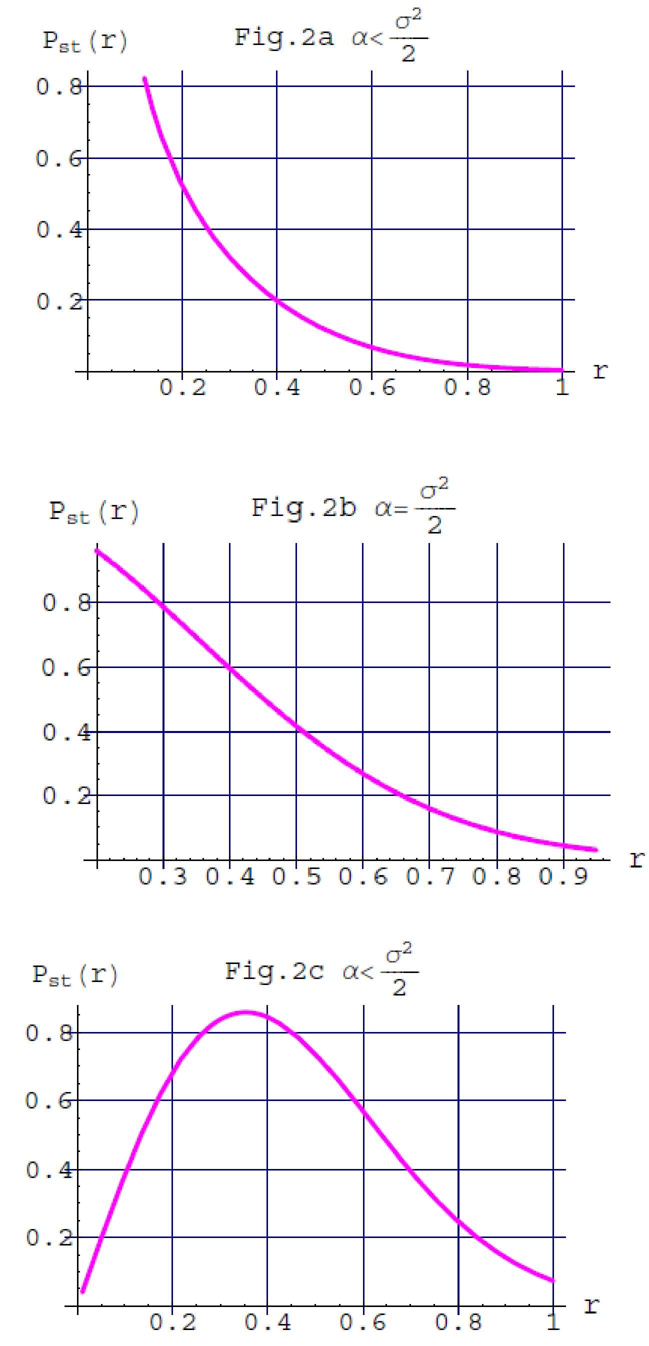

Fig.2. P-bifurcation of $p_\alpha(r)$ at $\alpha_p = \dfrac{\sigma^2}{2}$,where

$$\alpha = \mu_1 + \frac{\mu_2}{8}, \quad \sigma^2 = -\frac{\mu_4}{\mu_7}.$$

We now choose some values of the parameters in the equations, draw the graphics of $p_{st}(r)$. It is worth putting forward that calculating the Hopf bifurcation with the parameters in the original system is necessary. If we now have values of the original parameters in system (2.3), that $b = 1.5, \mu = 1, \bar{\omega} = 1, r_\alpha = 0.5625, \chi_\alpha = 0.19, \zeta_h = \zeta_\alpha = 0.3, \gamma_1 = 1.4, \kappa = 1.5, \delta_A = 0.2, B = 1.1, x_0 = 0.5, \sigma_1 = 0.3, \sigma_2 = 0.2.$ After further calculations we obtain $\mu_1 = -0.2, \mu_2 = 9.5907, \mu_3 = 0, \mu_4 = 5.1183,.$ $\mu_7 = -20.4732,$ then

$$p(r) = 0.28209 e^{-4r^2}.$$

What is more is that $\tilde{r} = 0.28209$ where $p_{st}(r)$ has the maximum value.

4.3 Case III: $\mu_3 \neq 0, \mu_7 \neq 0.$

When $\mu_3 \neq 0, \mu_7 \neq 0.$ Similar above Case II discussions, Through calculation, we can obtain

$$p_{st}(r) = c\exp\left(\frac{r^2\mu_7}{\mu_4}\right) r^2 \left(8\mu_3 + r^2\mu_4\right)^{-2+\frac{8\mu_1}{\mu_4}+\frac{\mu_2}{\mu_4}-\frac{8\mu_3\mu_7}{\mu_4^2}} \tag{4.8}$$

where c is a normalization constant, according to Namachivaya's theory, we calculate the most possible amplitude r^* of system (2.9), we have

$$r = \tilde{r} = \sqrt{\frac{-8\mu_1 - \mu_2 + \mu_4 - \sqrt{(8\mu_1 + \mu_2 - \mu_4)^2 - 32\mu_3\mu_7}}{2\mu_7}}.$$

In the meantime, $p_{st}(r)$ is 0 (minimum) at $r = 0$. This means that the system subjected to random excitations is almost unsteady at the equilibrium point ($r = 0$) in the meaning of probability. The conclusion is to go all the way with what has been obtained by the singular boundary theory. The original nonlinear stochastic system has a stochastic Hopf bifurcation at $r = \tilde{r}$.

$$x_1^2 + x_2^2 = \frac{-8\mu_1 - \mu_2 + \mu_4 - \sqrt{(8\mu_1 + \mu_2 - \mu_4)^2 - 32\mu_3\mu_7}}{2\mu_7}$$

$(i.e. \;\; r = \tilde{r})$.

We now choose some values of the parameters in the equations, draw the graphics of $p_{st}(r)$. It is worth putting forward that calculating the Hopf bifurcation with the parameters in the original system is necessary. If we now have values of the original parameters in system (2.6), that $b = 11, \mu = 1.2, \bar{\omega} = 1, r_\alpha = 0.5, \chi_\alpha = 0.18, \zeta_h = \zeta_\alpha = 0.2, \gamma_1 = 1, \kappa = 1.1, \delta_A = 0.2, B = 1, x_0 = 0.5, \sigma_1 = 0.1,$ $\sigma_2 = 0.3.$ After further calculations we obtain $\mu_1 = -0.4625,$ $\mu_2 = 18.9095, \mu_3 = 0.510106, \mu_4 = 12.284$, $\mu_7 = -0.3,$

$$p(r)=\frac{155.736e^{-0.227225r^2}x^2}{(2.68+0.990208r^2)^{4.40395}}.$$

What is more is that $\tilde{r}=0.162777$ where $p_{st}(r)$ has the maximum value.

ACKNOWLEDGMENT

This research was supported by the National Natural Sci-Science Foundation of China (No.11201089) and (No.11561009). Guangxi Natural Science Foundation(No. 2013GXNSFAA019014) and (No. 2013GXNSFBA019016).

REFERENCES

[1] V. Mukhopadhyay, Benchmark active control technology, Guid. Contr. Dynam. 23 (2000) 913-960.
[2] V. Mukhopadhyay, Flutter suppression control law design and testing for the active flexible wing, Aircraft 1 (1995) 45-51.
[3] G. Chen, J. Moiola, H. Wang, Bifurcation control: theories, methods, and applications, J. Bifur. Chaos 10 (2000) 511-548.
[4] M. Ramesh, S. Narayanan, Controlling chaotic motion in a two-dimensional airfoil using time-delayed feedback, Sou. Vibr. 5 (2001) 1037-1049.
[5] P. Yu, Y. Yuan, J. Xu, Study of double Hopf bifurcation and chaos for an oscillator with time delayed feedback, Inter. Commun. Nonl. Scien. Numer. Simul. 7 (2002) 69-91.
[6] Z. Huang, C. Chen, Random attractions and bifurcation for the classical Rayleigh-vander Pol equations with small noise, J. Math. Chem 51 (2013) 2340-2353.
[7] T. Zhang, Global analysis of continuous flow bioreactor and membrane reactor models with death and maintenance, J. Math. Chem. 50 (2012) 2239-2247.
[8] J. Vinals, F. Lepine, M. Gaudreault, Pitchfork and Hopf bifurcations in stochastic regulatory networks with delayed feedback, J. Biophy. 96 (2009) 305a.
[9] T. Caraballo, J. Langa, J. Robinson, A stochastic pitchfork bifurcation in a reactiondiffusion equation, R. Soc. Lond. 457 (2001) 2041-2061.
[10] L. Librescu, G. Chiocchia, P. Marzocca, Implications of cubic physical/aerodynamic nonlinearities on the character of the flutter instability boundary, Nonlin. Mech. 3 (2003) 173-199.
[11] P. Marzocca, L. Librescu. W. Silva, Flutter, post-flutter and control of a supersonic 2-D lifting surface, Guid. Contr. Dynam. 5 (2002) 962-970.
[12] Y. Yuan, P. Yu, Aeroelasticity of time-delayed feedback control of two-dimensional supersonic lifting surfaces, Guid. Contr. Dynam. 27 (2004) 795-803.
[13] L. Librescu, Aeroelastic stability of orthotropic heterogeneous thin panels in the vicinity of the flutter critical boundary-part I and part II, J. de M´ecanique 4 (1965) 51-76.
[14] D. Liu, Z. Yao, D. Sarhaddi, F. Chavez, From piston theory to a unified hypersonicsupersonic lifting surface method, Aircraft 34 (1997) 304-312.
[15] B. Thuruthimattam, P. Friedmann, J. McNamara, K. Powell, Aeroelasticity of a generic hypersonic vehicle, AIAA Paper 2002-2209, April 2002.
[16] Z. Chen, T. Zhang, Long time behaviour of a stochastic model for continuous flow bioreactor, J. Math. Chem. 51 (2013) 451-464.
[17] C. Chiarella, X. He, D. Wang, M. Zheng, The stochastic bifurcation behaviour of speculative financial markets, Physica A 387 (2008) 3837-3846.
[18] Y. Lin, G. Cai, Probabilistic Structural Dynamics, Mcgraw-hill Professional Publishing, 2004.
[19] H. Keller, Random attractors and and bifurcations of the stochastic Lorenz system, Technical Report 389, Institut f¨ur Dynamische Systeme, Universit¨at Bremen, 1996.
[20] D. Huang, H. Wang, Hopf bifurcation of the stochastic model on HAB nonlinear stochastic dynamics, Chaos, Solitons, Fractals 27 (2006) 1072-1079.
[21] L. Hong, J. Sun, Codimension two bifurcations of nonlinear systems driven by fuzzy noise, Physica D 213 (2006) 181-189.
[22] I. Chueshov, Monotone Random Systems Theory and Applications, Springer-verlag, 2002.
[23] Y. Kuznetsov, Elements of Applied Bfifurcation Theory, 3rd ed., Springer-Verlag New York. 2004.
[24] S. Chow, J. Hale, Methods of Bifurcation Theory, Springer-Verlag New York. 1982.
[25] H. Kielhofer, Bifurcation Theory An Introduction with Applications to PDEs, Springer-Verlag New York. 2004.
[26] L. Arnold, Random Dynamical Systems. Springer, New york, 1998.
[27] N. Sir Namachchivaya, Stochastic bifurcation, Appl. Math. Comput. 38 (1990) 101-159.
[28] W. Zhu, Nonlinear Stochastic Dynamics and Control in Hamiltonian Formulation, Beijing: Science Press, 2003.
[29] R. Khasminskii, On the principle of averaging for Ito' stochastic differential equations, Kybernetika(Prague) 4 (1968) 260-279.
[30] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
[31] J. Hale, S. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[32] M. Fofana, P. Ryba, Parametric stability of non-linear time delay equations, Nonlin. Mech. 39 (2004) 79-91.