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Solving Fuzzy Linear Systems

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***Abstract*—In paper the fuzzy linear equation $A\tilde{x}=\tilde{b}$ in which *A* is a crisp matrix and $\tilde{b}$ is an arbitrary fuzzy numbers vector, is investigated in detail. Numerical procedure for calculating the solution is designed and the sufficient condition for the existence of strong fuzzy solution is derived. Some examples are given to illustrate the proposed method.**

***Index Terms*—Fuzzy numbers, Matrix analysis, Fuzzy linear system, Fuzzy approximate solution.**

I. Introduction

System of simultaneous linear equations plays a major role in various areas such as mathematics, physics, statistics, engineering and social sciences. In many linear systems, some of the system parameters are vague or imprecise, and fuzzy mathematics is a better tool than crisp mathematics for modeling these problems, and hence how to design a better method for solving a fuzzy linear system is becoming more and more important thing. The concept of fuzzy numbers and their arithmetic operations were first introduced and investigated by Zadeh [29], Dubois et al.[15] and Nahmias [22]. A different approach to fuzzy numbers and the structure of fuzzy number spaces was given by Puri and Ralescu[26], Goetschell et al.[18] and Wu Congxin et al.[27, 29].

Since Friedman et al.[16] proposed a general model for solving an n×n fuzzy linear systems A $\tilde{x}=\tilde{b}$ by an embedding approach in 1998, lots of works have been done about some advanced fuzzy linear systems such as dual fuzzy linear systems (DFLS), general fuzzy linear systems (GFLS), fully fuzzy linear systems (FFLS)and general dual fuzzy linear systems (GDFLS) see [1−6, 9,13−14,30]. Some new theory and method for fuzzy linear systems still appeared [7, 10, 17, 19−20, 25−26] recently.

In this paper we propose a general model for solving the fuzzy linear system $A\tilde{x}=\tilde{b}$ by using a matrix approach. We extend the fuzzy linear $A\tilde{x}=\tilde{b}$ r system into a crisp system of linear equations. The fuzzy minimal solution of the fuzzy matrix equation is derived from solving the crisp function system. Moreover, the existence condition of the strong minimal fuzzy solution is discussed. Finally, some examples are given to illustrate our method.

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II. Preliminaries

There are several definitions for the concept of fuzzy numbers.

2.1. The fuzzy number

Definition 2.1.A fuzzy number is a fuzzy set like $u:R\to I=[0,1]$ which satisfies:

(1) *u*is upper semi-continuous,

(2) *u*is fuzzy convex,

i.e., $u(\lambda x+(1-\lambda)y)\ge\min\{u(x),u(y)\}$ for all $x,y\in R,\lambda\in[0,1]$,

(3) *u*is normal, i.e., there exists $x_0\in R$ such that $u(x_0)=1$,

(4) $\sup pu=\{x\in R|u(x)>0\}$ is the support of the*u*, and its closure cl(supp*u*) is compact.

Let E^1 be the set of all fuzzy numbers on*R*.

Definition 2.2. A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r),\overline{u}(r),0\le r\le 1$;which satisfies the requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

(3) $\underline{u}(r)\le\overline{u}(r),0\le r\le 1$.

A crisp number *x* is simply represented by $(\underline{u}(r),\overline{u}(r))=(x,x),0\le r\le 1$. By appropriate definitions the fuzzy number space $\{\underline{u}(r),\overline{u}(r)\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space.

Let $\tilde{x}=(\underline{x}(r),\overline{x}(r)),y=(\underline{y}(r),\overline{y}(r)),0\le r\le 1$, and $k\in R$. Then

(1) $\tilde{x}=\tilde{y}\ \ iff\ \ \underline{x}(r)=\underline{y}(r),\overline{x}(r)=\overline{y}(r)$,

(2) $\tilde{x}+\tilde{y}=(\underline{x}(r)+\underline{y}(r),\overline{x}(r)+\overline{y}(r))$,

(3) $\tilde{x}-\tilde{y}=(\underline{x}(r)-\overline{y}(r),\overline{x}(r)-\underline{y}(r))$,

(4) $k\tilde{x}=\begin{cases}(k\underline{x},k\overline{x}),k\ge 0\\(k\overline{x},k\underline{x}),k<0\end{cases}$,

Definition 2.3. A matrix $\tilde{A}=(\tilde{a}_{ij}),0\le i\le m,0\le j\le n$

is called a fuzzy matrix, if each element $\tilde{a}_{ij}$ of $\tilde{A}$ is a fuzzy number. Particularly, a vector $\tilde{x} = (\tilde{x}_j), 0 \le j \le n$ is called a fuzzy vector, if each element $\tilde{x}_j$ of $\tilde{x}$ is a fuzzy number

Definition 2.4. Let $A = (a_{ij})$ be a $m \times n$ crisp matrix and be a $\tilde{B} = (\tilde{b}_{ij}), 0 \le i \le n, 0 \le j \le p$ fuzzy matrix. The size of the product of two matrices is $m \times p$ and is written as follows:

$$A\tilde{B} = \tilde{C} = (\tilde{c}_{ij}),$$

where

$$\tilde{c}_{ij} = \sum_{k=1,\dots,n}^{+} a_{ik} \times \tilde{b}_{kj}.$$

Definition 2.5. The linear system equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_m \end{pmatrix}, \quad (2.1)$$

Where $a_{ij}, 0 \le i \le m, 0 \le j \le n$ are crisp numbers and $\tilde{b}_i, 1 \le i \le m$ are fuzzy numbers, is called a fuzzy linear system(FLS).

Using matrix notation, we have

$$A\tilde{x} = \tilde{b} \quad (2.2)$$

A fuzzy numbers vector

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^{\mathrm{T}}$$

is called a fuzzy solution of the fuzzy linear system if and only if $\tilde{x}$ satisfies (2.6).

III. SOLVING FUZZY LINEAR SYSTEMS

Definition 3.1. A fuzzy numbers vector $\tilde{u}$ in parametric form is a pair $(\bar{u}, \underline{u})$ of vector functions $\underline{u}(r), \bar{u}(r)$ where

$$\underline{u}(r) = (\underline{u}_1(r), \underline{u}_2(r), \dots, \underline{u}_n(r))^{\mathrm{T}},$$

$$\bar{u}(r) = (\bar{u}_1(r), \bar{u}_2(r), \dots, \bar{u}_n(r))^{\mathrm{T}},$$

$0 \le r \le 1$, which satisfies the requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous vector function,

(2) $\bar{u}(r)$ is a bounded monotonic decreasing left continuous vector function,

(3) $\underline{u}(r) \le \bar{u}(r), 0 \le r \le 1$.

Definition 3.2. Let $\tilde{u} = (\underline{u}(r), \bar{u}(r)), v = (\underline{v}(r), \bar{v}(r)) \in E^n, 0 \le r \le 1$ be two fuzzy vectorsand $k \in R$. Then

(1) $\tilde{u} = \tilde{v}$ iff $\underline{u}(r) = \underline{v}(r)$ and $\bar{u}(r) = \bar{v}(r)$

(2) $\tilde{u} + \tilde{v} = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$,

(3) $\tilde{u} - \tilde{v} = (\underline{u}(r) - \bar{v}(r), \bar{u}(r) - \underline{v}(r))$,

(4) $k\tilde{u} = \begin{cases} (k\underline{u}, k\bar{u}), k \ge 0. \\ (k\bar{u}, k\underline{u}), k < 0. \end{cases}$

Theorem 3.1. The fuzzy matrix equation (2.6) can be extended to a crisp function linear system as follows

$$Sx(r) = b(r) \quad (3.1)$$

where

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}, x(r) = \begin{pmatrix} \underline{x}(r) \\ -\bar{x}(r) \end{pmatrix}, b(r) = \begin{pmatrix} \underline{b}(r) \\ -\bar{b}(r) \end{pmatrix},$$

in which the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by the following way:

if $a_{ij} \ge 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \le i \le m, 1 \le i \le n$;

if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \le i \le m, 1 \le i \le n$.

Proof. We denote the right fuzzy vector of Eqs.(2.2) $\tilde{b}$ with $\tilde{b} = [\underline{b}(r), \bar{b}(r)], 0 \le r \le 1$ and the unknown fuzzy vector denoted by

$$\tilde{x} = [\underline{x}(r), \bar{x}(r)] = [\underline{x}_j(r), \bar{x}_j(r)]_{n\times 1}.$$

We also suppose $A = A^+ + A^-$ in which the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined bythe following way:

if $a_{ij} \ge 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \le i \le m, 1 \le i \le n$;

if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \le i \le m, 1 \le i \le n$.

For fuzzy linear equation $A\tilde{x} = \tilde{b}$, we can express it as

$$(A^+ + A^-)[\underline{x}(r), \bar{x}(r)] = [\underline{b}(r), \bar{b}(r)]. \quad (3.2)$$

Since

$$k\tilde{x}_j = \begin{cases} (k\underline{x}_j, k\bar{x}_j), k \ge 0 \\ (k\bar{x}_j, k\underline{x}_j), k < 0 \end{cases},$$

we have

$$A\tilde{x}_j = \begin{cases} (A\underline{x}_j, A\bar{x}_j), k \ge 0 \\ (A\bar{x}_j, A\underline{x}_j), k < 0 \end{cases},$$

so the Eqs.(3.2) be rewritten is

$$[A^+\underline{x}(r) + A^-\bar{x}(r), A^+\bar{x}(r) + A^-\underline{x}(r)] = [\underline{b}(r), \bar{b}(r)]$$

Thus we get

$$\begin{cases} A^+\underline{x}(r) + A^-\bar{x}(r) = \underline{b}(r) \\ A^+\bar{x}(r) + A^-\underline{x}(r) = \bar{b}(r) \end{cases} \quad (3.3)$$

or

$$\begin{cases} A^+\underline{x}(r) - A^-(-\bar{x}(r)) = \underline{b}(r) \\ -A^+\bar{x}(r) - A^-\underline{x}(r) = -\bar{b}(r) \end{cases}$$

Expressing Eqs.(3.3) in matrix form, we have

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} \underline{x}(r) \\ -\bar{x}(r) \end{pmatrix} = \begin{pmatrix} \underline{b}(r) \\ -\bar{b}(r) \end{pmatrix}. \quad \square$$

In order to solve the fuzzy linear equation (2.2), we need to consider the systems of linear equations (3.1). We consider a specious case, that is, the coefficient matrix A is a $n \times n$ matrix. By the M. Freidman[16], the following results are apparent.

Theorem 3.2. The matrix S is nonsingular if and only if the matrices $A^{+}+A^{-}$ and $A^{+}-A^{-}$ are both nonsingular. If S is nonsingular, then

$$S^{-1}=\frac{1}{2}\begin{pmatrix}(A^{+}-A^{-})^{-1}+(A^{+}+A^{-})^{-1} & (A^{+}-A^{-})^{-1}-(A^{+}+A^{-})^{-1}\\ (A^{+}-A^{-})^{-1}-(A^{+}+A^{-})^{-1} & (A^{+}-A^{-})^{-1}+(A^{+}+A^{-})^{-1}\end{pmatrix} \quad (3.4)$$

where $(A^{+}+A^{-})^{-1},(A^{+}-A^{-})^{-1}$ are inverse matrices of matrices $A^{+}+A^{-}$ and $A^{+}-A^{-}$,respectively. In this case, the solution of model equation (3.1) is

$$\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix}=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}^{-1}\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix} \quad (3.5)$$

By matrix theory analysis, we obtained the minimal solution[3] of the function linear system(3.1) as

$$x(r)=S^{\dagger}b(r),$$

i.e.

$$\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix}=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}^{\dagger}\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix}. \quad (3.6)$$

Where $S^{\dagger}$ is the Moore-Penrose generalized inverse of matrix *S*.

However, the solution matrix may still not be an appropriate fuzzy numbers vector. Restricting the discussion to triangular fuzzy numbers, i.e.,$\underline{b}_j(r),\overline{b}_j(r),1\le i\le n$ and consequently $\underline{x}_i(r),\overline{x}_i(r),1\le i\le n$ are all linear functions of *r*, and having calculated x(r) which solves (3.1),we define the fuzzy minimal solution to the fuzzy linear systems (2.2) as follows.

Definition 3.3. Let $x(r)=\underline{x}_i(r),\overline{x}_i(r),1\le i\le n$ be the minimal solution of (3.1). Thefuzzy number matrix $\underline{u}_i(r),\overline{u}_i(r),1\le i\le n$ defined by

$$\begin{cases}\underline{u}_i(r)=\min\{\underline{x}_i(r),\overline{x}_i(r),\underline{x}_i(1),\overline{x}_i(1)\},\\ \overline{u}_i(r)=\max\{\underline{x}_i(r),\overline{x}_i(r),\underline{x}_i(1),\overline{x}_i(1)\},\\ \quad 1\le i\le n,0\le r\le 1.\end{cases} \quad (3.7)$$

is called the fuzzy minimal solution of the function linear systems(3.1).If $\underline{x}_i(r),\overline{x}_i(r),1\le i\le n$ are all fuzzy numbers then $\underline{u}_i(r)=\underline{x}_i(r),\overline{u}_i(r)=\overline{x}_i(r),1\le i\le n$ and U is called a strong minimal fuzzy solution of the fuzzy linear systems (2.6). Otherwise, U is called a weak minimal fuzzy solution.

To illustrate the expression (3.6) to be a fuzzy solution vector, we now discuss the generalized inverses of non negative matrix

$$S=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}$$

in a special structure.

Lemma 3.1 [11]. Let

$$S=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}$$

Then the matrix

$$S^{\dagger}=\frac{1}{2}\begin{pmatrix}(A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger} & (A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger}\\ (A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger} & (A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger}\end{pmatrix}$$

(3.8)

is the Moore-Penrose inverse of the matrix S, where $(A^{+}-A^{-})^{\dagger},(A^{+}+A^{-})^{\dagger}$ are Moore-Penrose inverses of matrices $A^{+}+A^{-}$ and $A^{+}-A^{-}$,respectively.

The key points to make the solution matrix being a strong fuzzy solution is that $S^{\dagger}b(r)$ is fuzzymatrix, i.e., each element in which is a triangular fuzzy number. By the analysis, it is equivalent to the condition $S^{\dagger}\ge O$.

Theorem 3.3. If

$(A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger}\ge O,(A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger}\ge O,$

the fuzzy linear equation (2:6) has a strong fuzzy minimal solution as follows:

$$\tilde{x}=[\underline{x}(r),\overline{x}(r)]$$

where

$$\begin{cases}\underline{x}(r)=E\underline{b}(r)-\overline{b}(r),\\ \overline{x}(r)=-F\underline{b}(r)+E\overline{b}(r)\\ E=\frac{1}{2}((A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger}) \quad (3.9)\\ F=\frac{1}{2}((A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger})\end{cases}$$

Proof. Let

$$S^{\dagger}=\begin{pmatrix}E & F\\ F & E\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger} & (A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger}\\ (A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger} & (A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger}\end{pmatrix}$$

, we know the condition that $S^{+}\ge 0$ is equivalent to $E\ge O,F\ge O$.

Since $\tilde{b}=[\underline{b}(r),\overline{b}(r)],\underline{b}(r)$ is a bounded monotonic increasing left continuous function vector and $\overline{b}(r)$ a bounded monotonic decreasing left continuous function vector with $\underline{b}(r)\le\overline{b}(r),0\le r\le 1$by Definition 3.1.

According to Eqs.(3.6), we have

$$\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix}=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}^{\dagger}\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix},$$

i.e.,

$$\underline{x}(r)=E\underline{b}(r)-\overline{b}(r),$$
$$\overline{x}(r)=-F\underline{b}(r)+E\overline{b}(r).$$

Now that $E\ge O,F\ge O$ and $\underline{b}(r),-\overline{b}(r)$ are bounded monotonic increasing left continuous function vectors, we know that $\underline{x}(r)$ is a bounded monotonic increasing left continuous function vector and $\overline{x}(r)$ is a bounded monotonic decreasing left continuous function vector. And

$$\overline{x}(r)-\underline{x}(r)=E(\overline{b}(r)-\underline{b}(r))+F(\overline{b}(r)-\underline{b}(r))$$
$$=(E+F)(\overline{b}(r)-\underline{b}(r))=(A^{+}-A^{-})^{\dagger}(\overline{b}(r)-\underline{b}(r))\ge O.$$

Thus the fuzzy matrix equation (2.6) has a strong minimal fuzzy solution. □

The following Theorems give some results for such S^{-1} and $S^{\dagger}$ to be nonnegative. As usual, $(.)^{\mathrm{T}}$ denotes the transpose of a matrix (.).

Theorem 3.6[25]. The inverse S^{-1} of a nonnegative matrix S is nonnegative if and only if $S$is a generalized permutation matrix.

Theorem 3.7[12]. Let S be an $2n\times 2n$ nonnegative matrix with rank r. Then the following assertions are equivalent:

(a). $S^{\dagger}\geq 0$:

(b). There exists a permutation matrix P, such that PS has the form

$$PS=\begin{pmatrix} T_1\\ T_2\\ \vdots\\ T_r\\ O\end{pmatrix},$$

where each T_i has rank 1 and the rows of T_i are orthogonal to the rows of T_j, whenever$i\neq j$, thezero matrix may be absent.

$$(c).\ S^{\dagger}=\begin{pmatrix} GC^{\mathrm{T}} & GD^{\mathrm{T}}\\ GD^{\mathrm{T}} & GC^{\mathrm{T}}\end{pmatrix}$$

for some positive diagonal matrix G. In this case,

$$(C+D)^{\dagger}=G(C+D)^{\mathrm{T}},(C-D)^{\dagger}=G(C-D)^{\mathrm{T}}.$$

IV. NUMERICAL EXAMPLES

Example 4.1. Consider the following fuzzy linear system

$$\begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\begin{pmatrix}(r,2-r)\\ (4+r,7-2r)\end{pmatrix}.$$

Let

$$\tilde{x}=[\underline{x}(r),\overline{x}(r)]=\left[\begin{pmatrix}\underline{x}_1(r)\\ \underline{x}_2(r)\end{pmatrix},\begin{pmatrix}\overline{x}_1(r)\\ \overline{x}_2(r)\end{pmatrix}\right],A=A^{+}+A^{-}=\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}+\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}$$

and

$$\tilde{b}=[\underline{b}(r),\overline{b}(r)]=\left[\begin{pmatrix}r\\ 4+r\end{pmatrix},\begin{pmatrix}2-r\\ 7-2r\end{pmatrix}\right].$$

By the Theorem 3.1., the original fuzzy matrix equation is equivalent to the following function linear system

$$Sx(r)=b(r),$$

where

$$x(r)=\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix},b(r)=\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix}$$

and

$$S=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 1\\ 1 & 2 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 2\end{pmatrix}.$$

The matrix S is obviously invertible, its inverse is

$$S^{\dagger}=\begin{pmatrix}1.3333 & -0.3333 & 0.6667 & -0.6667\\ -0.6667 & 0.6667 & -0.3333 & 0.3333\\ 0.6667 & -0.6667 & 1.3333 & -0.3333\\ -0.3333 & 0.3333 & -0.6667 & 0.6667\end{pmatrix}.$$

From Eqs.(3.6), the solution of model is as follows:

$$\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix}=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}^{\dagger}\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix}$$

$$=\frac{1}{10}\begin{pmatrix}1.3333 & -0.3333 & 0.6667 & -0.6667\\ -0.6667 & 0.6667 & -0.3333 & 0.3333\\ 0.6667 & -0.6667 & 1.3333 & -0.3333\\ -0.3333 & 0.3333 & -0.6667 & 0.6667\end{pmatrix}\left[\begin{pmatrix}r\\ 4+r\end{pmatrix},\begin{pmatrix}-2+r\\ -7+2r\end{pmatrix}\right]$$

$$=\left[\begin{pmatrix}2.0000+0.3333r\\ 1.0000+0.3333r\end{pmatrix},\begin{pmatrix}-3.0000+0.6667r\\ -2.0000+0.6667r\end{pmatrix}\right]$$

i.e.,

$$\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\frac{1}{5}\begin{pmatrix}(2.0000+0.3333r,3.0000-0.6667r)\\ (1.0000+0.3333r,2.0000-0.6667r)\end{pmatrix}.$$

Since $\tilde{x}$ is an appropriate triangular fuzzy numbers vector, we obtained the solution of the fuzzy linear system is

$$\tilde{U}=\begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix}=\frac{1}{5}\begin{pmatrix}(2r,4-2r)\\ (-2+r,-r)\end{pmatrix}.$$

it admits a strong fuzzy solution by Definition 3.3.

Example 4.2. Consider the following fuzzy linear system

$$\begin{pmatrix}2 & -1\\ -2 & 1\end{pmatrix}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\begin{pmatrix}(r,2-r)\\ (-2+r,-r)\end{pmatrix}.$$

Let

$$\tilde{x}=[\underline{x}(r),\overline{x}(r)]=\left[\begin{pmatrix}\underline{x}_1(r)\\ \underline{x}_2(r)\end{pmatrix},\begin{pmatrix}\overline{x}_1(r)\\ \overline{x}_2(r)\end{pmatrix}\right],A=A^{+}+A^{-}=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & -1\\ -2 & 0\end{pmatrix}$$

and

$$\tilde{b}=[\underline{b}(r),\overline{b}(r)]=\left[\begin{pmatrix}r\\ -2+r\end{pmatrix},\begin{pmatrix}2-r\\ -r\end{pmatrix}\right].$$

By the Theorem 3.1., the original fuzzy matrix equation is equivalent to the following function linear system

$$Sx(r)=b(r),$$

where

$$x(r)=\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix},b(r)=\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix}$$

and

$$S=\begin{pmatrix}A^{+} & -A^{-}\\ -A^{-} & A^{+}\end{pmatrix}=\begin{pmatrix}2 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 2 & 2 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}.$$

The matrix S is obviously not invertible, its Moore-Penrose inverse is

$$S^{\dagger} = \frac{1}{10}\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \geq O.$$

From Eqs.(3.6), the minimal solution of model is as follows:

$$\begin{pmatrix} \underline{x}(r) \\ -\overline{x}(r) \end{pmatrix} = \begin{pmatrix} A^{+} & -A^{-} \\ -A^{-} & A^{+} \end{pmatrix}^{\dagger} \begin{pmatrix} \underline{b}(r) \\ -\overline{b}(r) \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\left[\begin{pmatrix} r \\ -2+r \end{pmatrix}, \begin{pmatrix} -2+r \\ r \end{pmatrix}\right]$$

$$= \frac{1}{5}\left[\begin{pmatrix} 2r \\ -1+2r \end{pmatrix}, \begin{pmatrix} -4+2r \\ r \end{pmatrix}\right]$$

i.e.,

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} (2r, 4-2r) \\ (-2+r, -r) \end{pmatrix}.$$

Since $\tilde{x}$ is an appropriate triangular fuzzy numbers vector, we obtained the solution of the fuzzy linear system is

$$\tilde{U} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} (2r, 4-2r) \\ (-2+r, -r) \end{pmatrix}.$$

it admits a strong fuzzy solution by Definition 3.3.

V. Conclusion

In this paper we proposed a general model for solving the fuzzy linear equation $A\tilde{x} = \tilde{b}$ where A is a crisp matrix and $\tilde{b}$ is an arbitrary fuzzy numbers matrix respectively by a matrix method. We extended the fuzzy linear system into a crisp system of linear equations and obtained the fuzzy minimal solution by solving the crisp function system. Moreover, the existence condition of the strong minimal fuzzy solution is discussed respectively. Numerical examples showed that our method is feasible to solve the fuzzy linear system.

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