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The 1-Good-Neighbor Diagnosability of Shuffle-Cubes

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Abstract—**The diagnosability is an important parameter in measuring the fault tolerance and the reliability of multiprocessor systems. In 2012, Peng et al. proposed a measure for fault diagnosis of the system, called the g-good-neighbor diagnosability that restrains each fault-free vertex containing at least g fault-free neighbors. The shuffle-cube SQ_n is a variation of the n-dimensional hypercube Q_n. In this paper, we show that the 1-good-neighbor diagnosability of SQ_n under the PMC model and MM* model is $2n-3$ for $n \geq 6$.**

Index Terms—**g-Good-neighbor diagnosability, Interconnection network, MM* model, PMC model, Shuffle-cube.**

I. Introduction

Interconnection networks (networks for short) as underlying topologies of many multiprocessor systems are usually represented by a graph where the vertices represent processors and the links represent communication links between processors. We use graphs and networks interchangeably. With a rapid increase in the number of processors in the multiple-processor system, the possibility that some processors may fail is rising. The reliability is one of the most important topics concerning the multiple-processor system and processors identification plays an essential role for reliable computing. The process of identifying the faulty processors is called the diagnosis of the system. A system G is said to be t-diagnosable if all faulty processors can be identified without replacement, provided that the number of faults presented does not exceed t. The diagnosability of G is the maximum value of t such that G is t-diagnosable [1]. For a t-diagnosable system, Dahbura and Masson in [2] proposed an algorithm with time complex $O(n^{2.5})$, which can effectively identify the set of faulty processors.

To identify the faulty processors, several diagnostic models were proposed in the prior work. One major approach is the PMC model introduced by Preparata et al. [3]. The diagnosis of the system is achieved through two linked processors testing each other. Another important model, called the MM model, was proposed by Maeng and Malek [4]. In the MM model, a vertex sends the same task to its two neighbors, and then compares their responses. The MM* is a specialization of the MM model in which each vertice must test any two of its neighbors. In 2005, Lai et al. introduced a restricted diagnosability called conditional diagnosability in [5]. They considered the situation that any faulty set cannot contain all the neighbors of any vertex in the system. In [6], Xu et al. showed that the conditional diagnosability of the n-dimensional shuffle-cube is $4n-15$ for $n \equiv 2\,(mod\ 4)$ and $n \geq 10$. In 2012, Peng et al. proposed a measure for faulty diagnosis of the system in [7], called the g-good-neighbor diagnosability (also called the g-good-neighbor conditional diagnosability), which requires that every fault-free vertice contains at least g fault-free neighbors. In [7],[8], they showed the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model and MM* model. In [9], Xu et al. showed the g-good-neighbor diagnosability of the complete cubic network under the PMC model and MM* model. Meanwhile, Yuan et al. in [10],[11] determined the g-good-neighbor diagnosability of the k-ary n-cube ($k \geq 3$ and $n \geq 3$) under the PMC model and MM* model. Since the probability that one fault-free vertex has at least one fault-free neighbor is much greater than the probability that one faulty vertex has at least one fault-free neighbor, the 1-good-neighbor diagnosability is more practical than the conditional diagnosability. Recently, in [12],[13], Wang et al. gave the g-good-neighbor diagnosability of Cayley graphs generated by the transposition trees under the PMC model and MM* model for $g = 1,2$. In [14], Zhao and Wang proved that the 1-good-neighbor diagnosability of augmented 3-ary n-cubes is $8n-10$ under the PMC model and MM* model for $n \geq 4$. In [15], Hao and Wang proved that the 1-good-neighbor diagnosability of augmented k-ary n-cubes under the PMC model and MM* model is $8n-9$ for $n \geq 4$ and $k \geq 4$. In [16], Jirimutu and Wang proved that the 1-good-neighbor diagnosability of alternating group graph networks is $2n-4$ for $n \geq 5$ under the PMC model and MM* model.

In this paper, we consider the 1-good-neighbor diagnosability of the shuffle-cube which is a new interconnection network topology presented by Li et al.[17]. The rest of the paper is organized as follows. Some preliminaries are provided in Section 2. Our main results are presented in Section 3. Finally, the conclusion is given in Section 4.

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II. PRELIMINARIES

In this section, some definitions and notations needed for our discussion, the PMC model and MM* model, and the n-dimensional shuffle-cube SQ_n are introduced.

A. Definitions and Notations

For convenience, the graphs and networks are used interchangeably. Given a nonempty vertex subset V' of V, the induced subgraph by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. The degree $d_G(u)$ of a vertex u is the number of edges incident with u. We use $\delta(G)$ to denote the minimum degrees of vertices of G, and $d_G(u,v)$ to denote the distance between u and v in G. For any vertex u, we define the neighborhood $N_G(u)$ of u in G to be the set of vertices adjacent to u. Let $S \subseteq V(G)$. We use $N_G(S)$ to denote the set $\bigcup_{u\in S} N_G(u) \setminus S$. For the neighborhoods and degrees, we usually omit the subscript for the graph G when no confusion arises. A graph G is k-regular if $d(u)=k$ for $u \in V(G)$. Let $G=(V,E)$ be a connected simple graph. The connectivity of G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left. A faulty set $F \subseteq V$ is called a g-good-neighbor faulty set if $|N(v)\cap(V \setminus F)| \geq g$ for $\forall v \in V \setminus F$. A g-good-neighbor cut of a graph G is a g-good-neighbor faulty set F such that $G-F$ is disconnected. The minimum cardinality of g-good-neighbor cuts is said to be the g-good-neighbor connectivity of G, denoted by $\kappa^{(g)}(G)$. For the terminology and notation which are not defined here, we follow [18].

B. The PMC Model and the MM* model

In the PMC model [10], each processor (vertex) in the faulty diagnosis system can perform tests on its neighbors. For two adjacent vertices u and v in $V(G)$, the ordered pair (u,v) represents the test performed by u on v. The test output is 1 or 0 which implies that the vertex being tested is faulty or fault-free. If the testing vertex is fault-free, then the test outputs are reliable; otherwise the outputs are unreliable. All the possible outputs of a test are shown in Table \ref{tab:test}. The result of all the tests is called a syndrome σ. For a given syndrome σ, a subset $F \subseteq V(G)$ is consistent with σ if the syndrome σ can be produced from the situation that, for any ordered pair (u,v) such that $u \in V \setminus F$, $\sigma(u,v)=1$ if and only if $v \in F$. Let $\sigma(F)$ denote all syndromes which F is consistent with. Under the PMC model, two distinct sets F_1 and F_2 of $V(G)$ are said to be indistinguishable if $\sigma(F_1)\cap\sigma(F_2)\neq\varnothing$; otherwise, F_1 and F_2 are said to be distinguishable. In other words, (F_1,F_2) is an indistinguishable pair if $\sigma(F_1)\cap\sigma(F_2)\neq\varnothing$; otherwise, (F_1,F_2) is a distinguishable pair.

Table 1. Output of test under the PMC model

Testing vertex	Tested vertex	Test output
fault-free	fault-free	0
fault-free	faulty	1
faulty	fault-free	0 or 1
faulty	faulty	0 or 1

In the MM model [10], a processor w sends the same task to the pair of distinct neighbors, u and v, and then compares their responses to diagnose a system G. The test, denoted by $(u,v)_w$, implies that u and v are adjacent to w, i.e., w can compare the responses from u and v. The MM* model is a special case of the MM model. If $w,u,v \in V(G)$ and $wu, wv \in E(G)$, then $(u,v)_w$ must be a test. All the possible outputs of a test are shown in Table \ref{tab:test2}. The collection of all the comparison results is called the syndrome σ^* of the diagnosis. For a given syndrome σ^*, a faulty set $F \subset V(G)$ is consistent with σ^* if and only if the following conditions are satisfied: 1. If $u,v \in F$ and $w \in V(G)\setminus F$, then $\sigma^*((u;v)_w)=1$; 2. If $u \in F$ and $v,w \in V(G)\setminus F$, then $\sigma^*((u,v)_w =1$; 3. If $u,v,w \in V(G)\setminus F$, then $\sigma^*((u,v)_w)=0$. The σ^* can be produced from F and all the vertices in $V \setminus F$ are fault-free. Since a faulty comparator can generate an unreliable result, a set of faulty vertices may produce different syndromes. Let $\sigma^*(F)$ denote all syndromes which F is consistent with. Similarly to the PMC model, two distinct sets F_1 and F_2 in $V(G)$ are said to be indistinguishable if $\sigma(F_1)\cap\sigma(F_2)\neq\varnothing$; otherwise, F_1 and F_2 are said to be distinguishable.

Table 2. Output of test under the MM* model

Testing vertex	Tested vertex	Test output
fault-free	fault-free, faulty-free	0
fault-free	faulty, fault-free(or faulty)	1
faulty	fault-free, fault-free	0 or 1
faulty	faulty, fault-free(or faulty)	0 or 1

A system $G=(V,E)$ is g-good-neighbor t-diagnosable if F_1 and F_2 are distinguishable, for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1|,|F_2|\leq t$. The g-good-neighbor diagnosability $t_g(G)$ of G is the maximum value of t such that G is g-good-neighbor t-diagnosable.

C. The n-dimensional shuffle-cube

As a variation of hypercubes Q_n, the n-dimensional shuffle-cube SQ_n, where $n \equiv 2\,(mod\ 4)$, is obtained from Q_n by changing some links. For SQ_n, the vertex set of SQ_n is represented by a set of n-bit binary string $u=u_{n-1}u_{n-2}\cdots u_1u_0$. Let $p_j(u)=u_{n-1}u_{n-2}\cdots u_{n-j}$ and $s_i(u)=u_{i-1}u_{i-2}\cdots u_1u_0$. The SQ_n is recursively defined as follows: SQ_2 is Q_2. For $n\geq 3$, SQ_n consists of 16 graphs SQ_{n-4}^{0000}, SQ_{n-4}^{0001}, $\cdots$, SQ_{n-4}^{1110} and SQ_{n-4}^{1111}, where $SQ_{n-4}^{i_1i_2i_3i_4}$ for

$i_j \in \{0,1\}$ and $1 \le j \le 4$ is obtained from SQ_{n-4} by adding a 4-bit binary string $i_1 i_2 i_3 i_4$ in the front of each vertex of SQ_{n-4} such that $V(SQ_{n-4}^{i_1 i_2 i_3 i_4}) = \{i_1 i_2 i_3 i_4 u_{n-5} \cdots u_1 u_0 : u_{n-5} \cdots u_1 u_0 \in V(SQ_{n-4})\}$ and $E(SQ_{n-4}^{i_1 i_2 i_3 i_4}) = \{(i_1 i_2 i_3 i_4 u_{n-5} \cdots u_1 u_0, i_1 i_2 i_3 i_4 v_{n-5} \cdots v_1 v_0) : (u_{n-5} \cdots u_1 u_0, v_{n-5} \cdots v_1 v_0) \in E(SQ_{n-4})\}$.

The vertices $u = u_{n-1} u_{n-2} \cdots u_1 u_0$ and $v = v_{n-1} v_{n-2} \cdots v_1 v_0$ in different $(n-4)$-dimensional graphs are linked by an edge in SQ_n if and only if $s_{n-4}(u) = s_{n-4}(v)$ and $p_4(u) \oplus p_4(v) \in V_{s_2(u)}$, where the symbol $\oplus$ denotes the addition with modulo 2 and $V_{00} = \{1111, 0001, 0010, 0011\}$, $V_{01} = \{0100, 0101, 0110, 0111\}$ $V_{10} = \{1000, 1001, 1010, 1011\}$, $V_{11} = \{1100, 1101, 1110, 1111\}$.

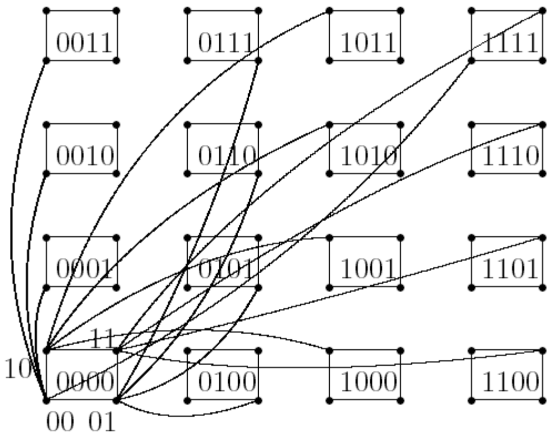

Figure 1. The Shuffle-cube SQ_6.

In Figure 1, we illustrate SQ_6 with edges only incident to the vertices of SQ_2^{0000}. As in [17], let $n = 4k+2$ and $u = u_{n-1} u_{n-2} \cdots u_1 u_0 = u_4^k u_4^{k-1} \cdots u_4^1 u_4^0$, where $u_4^0 = u_1 u_0$ and $u_4^j = u_{4j+1} u_{4j} u_{4j-1} u_{4j-2}$ for $1 \le j \le k$. Then the two vertices u and v of SQ_n are adjacent if and only if one of the following conditions holds:

(1) $u_4^{j^*} \oplus v_4^{j^*} \in V_{u_4^0}$ for exactly one j^* satisfying $1 \le j^* \le k$ and $u_4^j = v_4^j$ for all $0 \le j \ne j^* \le k$;

(2) $u_4^0 \oplus v_4^0 \in \{01, 10\}$ and $u_4^j = v_4^j$ for all $1 \le j \le k$.

Lemma 2.1. [17] SQ_n is n-regular and n-connected.

Lemma 2.2. [19] For $n \ge 6$, the 1-good-neighbor connectivity of SQ_n is 2n-4, i.e., $\kappa^{(1)}(SQ_n) = 2n-4$.

Lemma 2.3. [19] Let u and v be two adjacent vertices of SQ_n. If $u_4^0 = v_4^0 = 00$, then $|N(u) \cap N(v)| \le 2$. If $u_4^0 \ne 00$ or $v_4^0 \ne 00$, then $|N(u) \cap N(v)| = 0$.

Lemma 2.4. [6] Let u and v be two nonadjacent vertices of SQ_n such that $d(u,v) = 2$. If $u_4^0 \ne 00$ and $v_4^0 \ne 00$, then $|N(u) \cap N(v)| \le 4$. If $u_4^0 = 00$ or $v_4^0 = 00$, then $|N(u) \cap N(v)| \le 2$.

Lemma 2.5. [6] Let u and v be two nonadjacent vertices with $d(u,v) \ge 3$ in SQ_n, then $|N(u) \cap N(v)| = 0$.

Table 3. The number of common neighbors of any vertex pair u and v

	$d(u,v)=1$	$d(u,v)=2$
$u_4^0 = v_4^0 = 00$	$\lvert N(u) \cap N(v) \rvert \le 2$	$\lvert N(u) \cap N(v) \rvert \le 2$
$u_4^0 = 00$ $v_4^0 \ne 00$	$\lvert N(u) \cap N(v) \rvert = 0$	$\lvert N(u) \cap N(v) \rvert \le 2$
$u_4^0 \ne 00$ $v_4^0 \ne 00$	$\lvert N(u) \cap N(v) \rvert = 0$	$\lvert N(u) \cap N(v) \rvert \le 4$

III. THE 1-GOOD-NEIGHBOR DIAGNOSABILITY OF SHUFFLE-CUBES UNDER THE PMC MODEL AND MM* MDOEL

In this section, we shall prove the 1-good-neighbor diagnosability of shuffle-cubes SQ_n under the PMC model and MM* model. Let F_1 and F_2 be two distinct subsets of $V(G)$. We define the symmetric difference $F_1 \Delta F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. We first give two sufficient and necessary conditions for a system to be g-good-neighbor t-diagnosable under the PMC model and MM* model.

Theorem 3.1. [10] Under the PMC model, a system $G = (V, E)$ is g-good-neighbor t-diagnosable if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each distinct pair of g-good-neighbor faulty subsets F_1, F_2 of $V(G)$ with $|F_1|, |F_2| \le t$ (see Fig. 2).

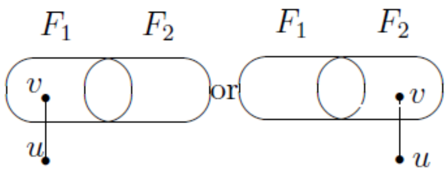

Figure 2. Illustration of a distinguishable pair (F_1, F_2) under the PMC model.

Theorem 3.2. [10]Under the MM * model, a system $G = (V, E)$ is g-good-neighbor t-diagnosable if and only if each distinct pair of two g-good-neighbor faulty subsets F_1 and F_2 of $V(G)$ with $|F_1|, |F_2| \le t$ satisfies one of the following conditions (see Fig. 3).

(1) There are two vertices $u, w \in V(G) \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \Delta F_2$ such that $uw \in E(G)$ and $vw \in E(G)$;

(2) There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw \in E(G)$ and $vw \in E(G)$;

(3) There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw \in E(G)$ and $vw \in E(G)$.

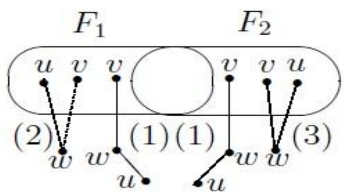

Fig.3. Illustration of a distinguishable pair (F_1, F_2) under the MM^* model.

Lemma 3.1. For $n \ge 6$, let $u = 0000\cdots000000 = u_4^k \cdots u_4^1 u_4^0$, $v = u_4^k \cdots u_4^2 (u_4^1 \oplus e) u_4^0$ where $e = 0001$, $A = \{u, v\}$, $F_1 = N(A)$ and $F_2 = A \cup N(A)$. Then $|F_1| = 2n-4$, $|F_2| = 2n-2$, $\delta(SQ_n - F_1) \ge 1$ and $\delta(SQ_n - F_2) \ge 1$.

Proof. By the definition of SQ_n, $uv \in E(SQ_n)$. Note that $e \in V_{u_4^0}$ and $u_4^0 = 00$. By Lemma 2.3, $|N(u) \cap N(v)| \le 2$. Since $N(u) \cap N(v) = \{0000\cdots001000, 0000\cdots001100\}$, $|N(u) \cap N(v)| = 2$. Thus, $|F_1| = |N(u),\ \{v\}| + |N(v),\ \{u\}| - |N(u) \cap N(v)| = n-1+n-1-2 = 2n-4$ and $|F_2| = |F_1| + |A| = 2n-2$. Next, we will prove $\delta(SQ_n - F_2) \ge 1$.

Case 1. $n = 6$.

Since $u = 000000$ and $v = 000100$, we have $F_1 = N(A) = \{000010, 000001, 001000, 001100, 111100, 000110, 000101, 111000\}$. It is easy to see that $SQ_6 - F_2$ is connected and $\delta(SQ_6 - F_2) \ge 1$. Combing this with that $\delta(SQ_6[A]) = 1$, we have $\delta(SQ_6 - F_1) \ge 1$.

Case 2. $n \ge 10$.

Let w be an arbitrary vertex of $SQ_n - F_2$. By Lemma 2.4, $|N(w) \cap N(u)| \le 4$ and $|N(w) \cap N(v)| \le 4$. Then $|N(w) \cap F_1| \le 8$. Thus, $|N_{SQ_n - F_2}(w)| \ge n - 8 \ge 2$ for $n \ge 10$. This means $\delta(SQ_n - F_2) \ge 1$. Combing this with that $\delta(SQ_n[A]) = 1$, $\delta(SQ_n - F_1) \ge 1$.

The proof is complete.

Lemma 3.2. For $n \ge 6$, the 1-good-neighbor diagnosability of SQ_n under the PMC model and MM* model is less than $2n-2$, i.e., $t_1(SQ_n) \le 2n-3$.

Proof. Let A, F_1 and F_2 be defined as Lemma 3.1. By Lemma 3.1, $|F_1| = 2n-4$, $|F_2| = 2n-2$, $\delta(SQ_n - F_1) \ge 1$ and $\delta(SQ_n - F_2) \ge 1$. Thus, F_1 and F_2 are both 1-good-neighbor faulty sets of SQ_n with $|F_1|, |F_2| \le 2n-2$. Since $A = F_1 \Delta F_2$ and $F_1 = N(A) \subset F_2$, there are no edges of SQ_n between $V(SQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Theorem 3.1 and Theorem 3.2, we can deduce that SQ_n is not 1-good-neighbor $(2n-2)$-diagnosable under the PMC model and MM* model. Therefore, by the definition of the 1-good-neighbor diagnosability, we can deduce that the 1-good-neighbor diagnosability of SQ_n under the PMC model and MM* model is less than $2n-2$, i.e., $t_1(SQ_n) \le 2n-3$.

The proof is complete.

Lemma 3.3. For $n \ge 6$, the 1-good-neighbor diagnosability of SQ_n under the PMC model is greater than or equal to $2n-3$, i.e., $t_1(SQ_n) \ge 2n-3$.

Proof. By the definition of the 1-good-neighbor diagnosability, it is sufficient to prove that SQ_n is 1-good-neighbor $(2n-3)$-diagnosable. To prove the statement, by Theorem \ref{T-g-good-t-diag}, it is equivalent to show that there exist two vertices $u \in V(SQ_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ such that $uv \in E(SQ_n)$ for each distinct pair of 1-good-neighbor faulty subsets F_1 and F_2 of $V(SQ_n)$ with $|F_1|, |F_2| \le 2n-3$.

The proof proceeds by way of contradiction. By Theorem 3.1, we suppose that there are two distinct 1-good-neighbor faulty subsets F_1 and F_2 of $V(SQ_n)$ with $|F_1|, |F_2| \le 2n-3$ such that there are no edges between $V(SQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. Without loss of generality, we assume that $F_2 \setminus F_1 \ne \emptyset$. Suppose that $V(SQ_n) = F_1 \cup F_2$. By the definition of SQ_n, $|V(SQ_n)| = 2^n$. Then $2^n = |F_1 \cup F_2| \le |F_1| + |F_2| = 2n-3+2n-3 = 4n-6$. Since $2^n > 4n-6$ for $n \ge 6$, this is a contradiction. Therefore, $V(SQ_n) \ne F_1 \cup F_2$.

Since there are no edges between $V(SQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, $SQ_n - F_1$ has two parts $SQ_n - F_1 - F_2$ and $SQ_n[F_2 \setminus F_1]$ (for convenience). Combing this with that F_1 is a 1-good-neighbor faulty set, $\delta(SQ_n - F_1 - F_2) \ge 1$, $\delta(SQ_n[F_2 \setminus F_1]) \ge 1$, and $|F_2 \setminus F_1| \ge 2$. Similarly, $\delta(SQ_n[F_1 \setminus F_2]) \ge 1$ when $F_1 \setminus F_2 \ne \emptyset$. Thus, $F_1 \cap F_2$ is a 1-good-neighbor faulty set. Since there are no edges between $V(SQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, we have that $F_1 \cap F_2$ is a 1-good-neighbor cut. By Theorem 2.2, $|F_1 \cap F_2| \ge 2n-4$. Then $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \ge 2 + 2n - 4 = 2n-2$, contradicting the supposition that $|F_2| \le 2n-3$. Therefore, SQ_n is 1-good-neighbor $(2n-3)$-diagnosable, i.e., $t_1(SQ_n) \ge 2n-3$.

The proof is complete.

Combining Lemma 3.2 and Lemma 3.3, we have the following theorem.

Theorem 3.3 For $n \ge 6$, the 1-good-neighbor diagnosability of SQ_n under the PMC model is $2n-3$, i.e., $t_1(SQ_n) = 2n-3$.

Lemma 3.4. For $n \ge 6$, the 1-good-neighbor diagnosability of SQ_n under the MM* model is greater than or equal to $2n-3$, i.e., $t_1(SQ_n) \ge 2n-3$.

Proof. By the definition of the 1-good-neighbor diagnosability, it is sufficient to show that SQ_n is 1-good-neighbor $(2n-3)$-diagnosable. By Theorem 3.2, suppose, on the contrary, that there are two distinct 1-good-neighbor faulty subsets F_1 and F_2 of SQ_n with $|F_1|, |F_2| \le 2n-3$, and the vertex set pair (F_1, F_2) does not satisfy with any one condition in Theorem 3.2. Without loss of generality, we assume that $F_2 \setminus F_1 \neq \emptyset$. A similar argument on . $V(SQ_n) \neq F_1 \cup F_2$. in Lemma 3.3 can be used to establish that $V(SQ_n) \neq F_1 \cup F_2$. Therefore, $V(SQ_n) \neq F_1 \cup F_2$.

Claim 1. $SQ_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $SQ_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a 1-good-neighbor faulty set, there is at least one vertex $u \in F_2 \setminus F_1$ such that u and w are adjacent. Note that the vertex set pair (F_1, F_2) does not satisfy with any one condition in Theorem 3.2. By the condition (3) of Theorem 3.2, there is at most one vertex $u \in F_2 \setminus F_1$ such that u and w are adjacent. Thus, there is just one vertex $u \in F_2 \setminus F_1$ such that u and w are adjacent. Similarly, there is just one vertex $v \in F_1 \setminus F_2$ such that v and w are adjacent. Clearly, $F_1 \setminus F_2 \neq \emptyset$. Let $W \subseteq V(SQ_n) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $SQ_n[V(SQ_n) \setminus (F_1 \cup F_2)]$ and H be the induced subgraph by the vertex set $V(SQ_n) \setminus (F_1 \cup F_2 \cup W)$. For any arbitrary vertex $w \in W$, there are $(n-2)$ neighbors in $F_1 \cap F_2$. Since $|F_2| \leq 2n-3$, $\sum_{w \in W} |N_{SQ_n[F_1 \cap F_2]}(w)| = |W|(n-2) \leq \sum_{v \in F_1 \cap F_2} d_{SQ_n}(v) \leq n|F_1 \cap F_2|$ $\leq n(|F_2|-1) \leq n(2n-4)$. Then $|W| \leq 2n$. We assume that $V(H) = \emptyset$. Then $|V(SQ_n)| = |F_1 \cup F_2| + |W| \leq |F_1| + |F_2|$ $+ |W| \leq 2(2n-3) + 2n = 6n-6 < 2^n$ for $n \geq 6$, which is a contradiction to $|V(SQ_n)| = 2^n$. So $V(H) \neq \emptyset$.

Since H contains no isolated vertex and (F_1, F_2) does not satisfy with the condition (1) of Theorem 3.2, there are no edges between $V(H)$ and $F_1 \Delta F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of SQ_n. Since F_1 is a 1-good-neighbor faulty set of SQ_n, we have that $\delta(H) \geq 1$ and $\delta(SQ_n[W \cup (F_2 \setminus F_1)]) \geq 1$. Similarly, F_2 is a 1-good-neighbor faulty set of SQ_n, we also have $\delta(SQ_n[W \cup (F_1 \setminus F_2)]) \geq 1$. Then $\delta(SQ_n[W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)]) \geq 1$. Note that $SQ_n - (F_1 \cap F_2)$ has two parts (for convenience): H and $SQ_n[W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)]$. Therefore, $F_1 \cap F_2$ is a 1-good-neighbor vertex cut of SQ_n. By Lemma 2.2, $|F_1 \cap F_2| \geq 2n-4$. Since $|F_1|, |F_2| \leq 2n-3$, $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $u \in F_2 \setminus F_1$ and $v \in F_1 \setminus F_2$. By Lemma 2.3 and Lemma 2.4, $|W| \leq |N(u) \cap N(v)| \leq 4$. We consider the following cases.

Case 1. $3 \leq |W| \leq 4$.

Let $\{w_1, w_2, w_3\} \subseteq W$. Then $|N_{SQ_n[F_1 \cap F_2]}(W)| \geq |N_{SQ_n[F_1 \cap F_2]}(\{w_1, w_2, w_3\})|$. Suppose that u, v are adjacent. By Lemma 2.3, u, v have at most two common neighbors. This means $|W| \leq 2$, which is a contradiction to $3 \leq |W| \leq 4$. So u, v are nonadjacent. Clearly, $N(u) \setminus \{w_1, w_2, w_3\} \subseteq F_1 \cap F_2$, $N(v) \setminus \{w_1, w_2, w_3\} \subseteq F_1 \cap F_2$, $N(w_1) \setminus \{u, v\} \subseteq F_1 \cap F_2$, $N(w_2) \setminus \{u, v\} \subseteq F_1 \cap F_2$ and $N(w_3) \setminus \{u, v\} \subseteq F_1 \cap F_2$. Suppose that $u_4^0 = 00$ or $v_4^0 = 00$. By Lemma 2.4, $|N(u) \cap N(v)| \leq 2$. This means $|W| \leq 2$, which is a contradiction to $3 \leq |W| \leq 4$. So $u_4^0 \neq 00$ and $v_4^0 \neq 00$. By Lemma 2.4, $|N(u) \cap N(v)| \leq 4$, $|N(w_1)| \cap |N(w_2)| \leq 4$, $|N(w_1)| \cap |N(w_3)| \leq 4$, and $|N(w_2)| \cap |N(w_3)| \leq 4$. Since w_1, w_2, w_3 are three common neighbors of u, v , we have $|N_{SQ_n[F_1 \cap F_2]}(u) \cap N_{SQ_n[F_1 \cap F_2]}(v)| \leq 1$. By Lemma 2.3, we have $|N(u) \cap N(w_i)| = 0$ and $|N(v) \cap N(w_i)| = 0$ for $i = 1, 2, 3$. Then

$$
\begin{aligned}
|F_1 \cap F_2| \geq{} & |N(u) \setminus \{w_1, w_2, w_3\}| + |N(v) \setminus \{w_1, w_2, w_3\}| + \\
& |N(w_1) \setminus \{u, v\}| + |N(w_2) \setminus \{u, v\}| + |N(w_3) \setminus \{u, v\}| \\
& - |N_{SQ_n[F_1 \cap F_2]}(u) \cap N_{SQ_n[F_1 \cap F_2]}(v)| - \sum_{1 \leq i \leq 3} |N(u) \cap N(w_i)| \\
& - \sum_{1 \leq i \leq 3} |N(v) \cap N(w_i)| - |N_{SQ_n[F_1 \cap F_2]}(w_1) \cap N_{SQ_n[F_1 \cap F_2]}(w_2)| \\
& - |N_{SQ_n[F_1 \cap F_2]}(w_1) \cap N_{SQ_n[F_1 \cap F_2]}(w_3)| - |N_{SQ_n[F_1 \cap F_2]}(w_2) \cap \\
& N_{SQ_n[F_1 \cap F_2]}(w_3)| \geq 2 \times (n-3) + 3 \times (n-2) - 1 - 0 - 0 - 3 \times 2 \\
={} & 5n - 19
\end{aligned}
$$

Thus, $|F_1| = |F_1 \cap F_2| + |F_1 \setminus F_2| \geq 5n-19+1 = 5n-18 > 2n-3$ for $n \geq 6$, a contradiction to $|F_1| \leq 2n-3$.

Case 2. $|W| = 2$.

Let $W = \{w_1, w_2\}$. Then $w_1 v, w_1 u, w_2 v, w_2 u \in E(SQ_n)$.

Case 2.1. u, v are adjacent.

Clearly, $N(u) \setminus \{w_1, w_2, v\} \subseteq F_1 \cap F_2$, $N(v) \setminus \{w_1, w_2, u\} \subseteq F_1 \cap F_2$, $N(w_1) \setminus \{u, v\} \subseteq F_1 \cap F_2$, and $N(w_2) \setminus \{u, v\} \subseteq F_1 \cap F_2$.

Suppose that $u_4^0 \neq 00$ or $v_4^0 \neq 00$. By Lemma 2.3, $|N(u) \cap N(v)| = 0$. This means $|W| = 0$, which is a contradiction to $|W| = 2$. So $u_4^0 = 00$ and $v_4^0 = 00$. Suppose that $[w_1]_4^0 \neq 00$. By Lemma 2.3, $|N(u) \cap N(w_1)| = 0$. Since v is a common neighbor of u and w_1, this is a contradiction. So $[w_1]_4^0 = 00$. Similarly, $[w_2]_4^0 = 00$. By Lemma 2.3, $|N(u) \cap N(v)| \leq 2$, $|N(u) \cap N(w_1)| \leq 2$, $|N(u) \cap N(w_2)| \leq 2$, $|N(v) \cap N(w_1)| \leq 2$, and $|N(v) \cap N(w_2)| \leq 2$. Since w_1, w_2 are two common neighbors of u and v, $|N_{SQ_n[F_1 \cap F_2]}(u) \cap N_{SQ_n[F_1 \cap F_2]}(v)| = 0$. By using the similar argument, we have $|N_{SQ_n[F_1 \cap F_2]}(w_1) \cap N_{SQ_n[F_1 \cap F_2]}(u)| \leq 1$, $|N_{SQ_n[F_1 \cap F_2]}(w_1) \cap N_{SQ_n[F_1 \cap F_2]}(v)| \leq 1$, $|N_{SQ_n[F_1 \cap F_2]}(w_2) \cap N_{SQ_n[F_1 \cap F_2]}(u)| \leq 1$, and $|N_{SQ_n[F_1 \cap F_2]}(w_2) \cap N_{SQ_n[F_1 \cap F_2]}(v)| \leq 1$. By Lemma 2.4, $|N(w_1) \cap N(w_2)| \leq 2$. Since u, v are two common neighbors of w_1 and w_2, $|N_{SQ_n[F_1 \cap F_2]}(w_1) \cap N_{SQ_n[F_1 \cap F_2]}(w_2)| = 0$. Then $|F_1 \cap F_2|$

$\geq |N(u) \setminus \{w_1,w_2,v\}| + |N(v) \setminus \{w_1,w_2,u\}| + |N(w_1) \setminus \{u,v\}| + |N(w_2) \setminus \{u,v\}| - |N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| \geq 2(n-3)+2(n-2)-0-4\times 1-0 = 4n-14$. Thus, $|F_1| = |F_1\cap F_2| + |F_1 \setminus F_2| \geq 4n-14+1 = 4n-13 > 2n-3$ for $n\geq 6$, a contradiction to $|F_1| \leq 2n-3$.

Case 2.2. u,v are nonadjacent.

Obviously, $N(u) \setminus \{w_1,w_2\} \subseteq F_1\cap F_2$, $N(v) \setminus \{w_1,w_2\} \subseteq F_1 \cap F_2$, $N(w_1) \setminus \{u,v\} \subseteq F_1\cap F_2$, and $N(w_2) \setminus \{u,v\} \subseteq F_1 \cap F_2$.

Case 2.2.1. $n=6$.

Case 2.2.1.1. $u_4^0 = v_4^0 = 00$.

Since $n=6$ and $u_4^0 = v_4^0 = 00$, we have $u_4^1 \neq v_4^1$. Suppose that $[w_1]_4^0 \neq 00$. Since $uw_1 \in E(SQ_6)$, by the definition of SQ_n, $u_4^0 \oplus [w_1]_4^0 \in \{01,10\}$ and $u_4^1 = [w_1]_4^1$. Similarly, $v_4^1 = [w_1]_4^1$. Thus, we have $u_4^1 = v_4^1$, which is a contradiction to $u_4^1 \neq v_4^1$. So $[w_1]_4^0 = 00$. Similarly, $[w_2]_4^0 = 00$. Meanwhile, $[w_1]_4^1 = u_4^1 \oplus e_1 = v_4^1 \oplus e_2$, $[w_2]_4^1 = u_4^1 \oplus e_3 = v_4^1 \oplus e_4$, where $e_1,e_2,e_3,e_4 \in V_{00}$. Since w_1,w_2 are nonadjacent, we have $[w_1]_4^1 \oplus [w_2]_4^1 \notin V_{00}$, i.e., $(u_4^1\oplus e_1) \oplus (u_4^1 \oplus e_3) \notin V_{00}$. Since $u_4^1 \oplus u_4^1 = 0000$, we have $e_1 \oplus e_3 \notin V_{00}$. Therefore, either $e_1 = 1111$ or $e_3 = 1111$. Similarly, either $e_2 = 1111$ or $e_4 = 1111$. Without loss of generality, we assume that $e_1 = 1111$. Since $[w_1]_4^1 = u_4^1 \oplus e_1 = v_4^1 \oplus e_2$ and $u_4^1 \neq v_4^1$, we have $e_2 \neq e_1$. This means $e_2 \neq 1111$. Then $e_4 = e_1 = 1111$ and $e_2, e_3 \in \{0001, 0010, 0011\}$. Thus, $w_1 = (u_4^1 \oplus 1111)00$ and $w_2 = (v_4^1 \oplus 1111)00$. By the definition of SQ_n, it is easy to see that $|N(w_1)| \cap N(u)| = 0$ and $|N(w_2)| \cap N(v)| = 0$. Since $w_1 = (v_4^1 \oplus e_2)00$ and $w_2 = (u_4^1 \oplus e_3)00$, we have $N(w_1) \cap N(v) = \{(v_4^1 \oplus e_i)00), e_i \in \{0001,0010,0011\} \setminus \{e_2\}\}$, and $N(w_2)\cap N(u) = \{(u_4^1 \oplus e_i)00), e_i \in \{0001,0010,0011\} \setminus \{e_3\}\}$, i.e., $|N(w_1) \cap N(v)| = 2$ and $|N(w_2) \cap N(u)| = 2$. By Lemma \ref{uv-nonadja-neighbor}, $|N(u) \cap N(v)| \leq 2$. Since w_1, w_2 are two common neighbors of u and v, $|N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| = 0$. Similarly, $|N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| = 0$. Then $|F_1 \cap F_2| \geq |N(u) \setminus \{w_1,w_2\}| + |N(v) \setminus \{w_1,w_2\}| + |N(w_1) \setminus \{u,v\}| + |N(w_2) \setminus \{u,v\}| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| \geq 4\times(n-2)-2-0-2-0-0 = 4\times n-12$. Thus, $|F_1| = |F_1\cap F_2| + |F_1 \setminus F_2| \geq 4n-12+1 = 4n-11 > 2n-3$ for $n=6$, which is a contradiction to $|F_1| \leq 2n-3$.

Case 2.2.1.2. $u_4^0 = 00$ and $v_4^0 \neq 00$.

By Lemma 2.4, $|N(u)\cap N(v)| \leq 2$ and $|N(w_1) \cap N(w_2)| \leq 4$. Since w_1, w_2 are two common neighbors of u and v, $|N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| = 0$. Similarly, $|N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| \leq 2$. By Lemma 2.3, $|N(u)\cap N(w_1)| \leq 2$, $|N(u) \cap N(w_2)| \leq 2$, $|N(v) \cap N(w_1)| = 0$, and $|N(v) \cap N(w_2)| = 0$. Thus, $|F_1\cap F_2| \geq |N(u) \setminus \{w_1,w_2\}| + |N(v) \setminus \{w_1,w_2\}| + |N(w_1) \setminus \{u,v\}| + |N(w_2) \setminus \{u,v\}| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| \geq 4\times(n-2)-2-0-2-0-0-2 = 4n-14$. Thus, $|F_1| = |F_1\cap F_2| + |F_1 \setminus F_2| \geq 4n-14+1 = 4n-13 > 2n-3$ for $n=6$, which is a contradiction to $|F_1| \leq 2n-3$.

Case 2.2.1.3. $u_4^0 \neq 00$ and $v_4^0 \neq 00$.

By Lemma 2.3, $|N(u) \cap N(w_1)| = 0$, $|N(u) \cap N(w_2)| = 0$, $|N(v) \cap N(w_1)| = 0$, and $|N(v)\cap N(w_2)| = 0$. By Lemma 2.4, $|N(u) \cap N(v)| \leq 4$. Since w_1, w_2 are two common neighbors of u and v, $|N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| \leq 2$. Similarly, $|N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| \leq 2$. Then $|F_1 \cap F_2| \geq |N(u) \setminus \{w_1,w_2\}| + |N(v) \setminus \{w_1,w_2\}| + |N(w_1) \setminus \{u,v\}| + |N(w_2) \setminus \{u,v\}| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| - |N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| \geq 4\times(n-2)-4\times 0-2\times 2 = 4n-12$. Thus, $|F_1| = |F_1 \cap F_2| + |F_1 \setminus F_2| \geq 4n-12+1 = 4n-11 > 2n-3$ for $n=6$, which is a contradiction to $|F_1| \leq 2n-3$.

Case 2.2.2. $n\geq 10$.

By Lemma 2.3, $|N(w_1) \cap N(u)| \leq 2$, $|N(w_1)\cap N(v)| \leq 2$, $|N(w_2) \cap N(u)| \leq 2$, and $|N(w_2) \cap N(v)| \leq 2$. By Lemma 2.4, $|N(u) \cap N(v)| \leq 4$ and $|N(w_1) \cap N(w_2)| \leq 4$. Since u,v are two common neighbors of w_1 and w_2, we have that $|N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(w_2)| \leq 2$. Similarly, $|N_{SQ_n[F_1\cap F_2]}(u) \cap N_{SQ_n[F_1\cap F_2]}(v)| \leq 2$. Then $|F_1 \cap F_2| \geq |N(u) \setminus \{w_1,w_2\}| + |N(v) \setminus \{w_1,w_2\}| + |N(w_1) \setminus \{u,v\}| + |N(w_2) \setminus \{u,v\}| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(v)| - |N_{SQ_n[F_1\cap F_2]}(w_1) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(u)| - |N_{SQ_n[F_1\cap F_2]}(w_2) \cap N_{SQ_n[F_1\cap F_2]}(v)|$

$-|N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(w_2)|\ge 4(n-2)-4\times 2-2\times 2=4n-20$.

Thus, $|F_1|=|F_1\cap F_2|+|F_1\setminus F_2|\ge 4n-20+1=4n-19>2n-3$ for $n\ge 10$, which is a contradiction to $|F_1|\le 2n-3$.

Case 3. $|W|=1$.

Let $W=\{w_1\}$. Then $w_1u, w_1v\in E(SQ_n)$.

Case 3.1. u,v are adjacent.

Clearly, $N(u)\setminus\{w_1,v\}\subseteq F_1\cap F_2$, $N(v)\setminus\{w_1,u\}\subseteq F_1\cap F_2$, and $N(w_1)\setminus\{u,v\}\subseteq F_1\cap F_2$. By Lemma 2.3, $|N(u)\cap N(v)|\le 2$. Since w_1 is one common neighbor of u and v, $|N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)|\le 1$. Similarly, $|N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(w_1)|\le 1$ and $|N_{SQ_n[F_1\cap F_2]}(v)\cap N_{SQ_n[F_1\cap F_2]}(w_1)|\le 1$. Then $|F_1\cap F_2|\ge|N(u)\setminus\{w_1,v\}|+|N(v)\setminus\{w_1,u\}|+|N(w_1)\setminus\{u,v\}|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(v)|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(u)|-|N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)|\ge 3(n-2)-3\times 1=3n-9$. Thus, $|F_1|=|F_1\cap F_2|+|F_1\setminus F_2|\ge 3n-9+1=3n-8>2n-3$ for $n\ge 6$, a contradiction to $|F_1|\le 2n-3$.

Case 3.2. u,v are nonadjacent.

Clearly, $N(u)\setminus\{w_1\}\subseteq F_1\cap F_2$, $N(v)\setminus\{w_1\}\subseteq F_1\cap F_2$, and $N(w_1)\setminus\{u,v\}\subseteq F_1\cap F_2$.

Case 3.2.1. $u_4^0=00$ and $v_4^0=00$.

By Lemma \ref{uv-nonadja-neighbor}, $|N(u)\cap N(v)|\le 2$. Since w_1 is one common neighbor of u,v, $N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)\le 1$. By Lemma 2.3, $|N_{SQ_n[F_1\cap F_2]}(w_1)\cap|N_{SQ_n[F_1\cap F_2]}(u)|\le 2$. Similarly, $|N_{SQ_n[F_1\cap F_2]}(w_1)\cap|N_{SQ_n[F_1\cap F_2]}(v)|\le 2$. Then $|F_1\cap F_2|\ge|N(u)\setminus\{w_1\}|+|N(v)\setminus\{w_1\}|+N(w_1)\setminus\{u,v\}-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(u)|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(v)|-|N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)|\ge n-1+n-1+n-2-2-2-1=3n-9$. Thus, $|F_1|=|F_1\cap F_2|+|F_1\setminus F_2|\ge 3n-9+1=3n-8>2n-3$ for $n\ge 6$, a contradiction to $|F_1|\le 2n-3$.

Case 3.2.2. $u_4^0=00$ and $v_4^0\ne 00$.

By Lemma \ref{uv-nonadja-neighbor}, $N(u)\cap N(v)\le 2$. Since w_1 is one common neighbor of u and v, $N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)\le 1$. By Lemma 2.3, $|N(u)\cap N(w_1)|\le 2$ and $|N(v)\cap N(w_1)|=0$. Then $|F_1\cap F_2|\ge|N(u)\setminus\{w_1\}|+|N(v)\setminus\{w_1\}|+|N(w_1)\setminus\{u,v\}|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(u)|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap N_{SQ_n[F_1\cap F_2]}(v)|-|N_{SQ_n[F_1\cap F_2]}(u)\cap|N_{SQ_n[F_1\cap F_2]}(v)|\ge n-1+n-1+n-2-0-2-1=3n-7$. Thus, $|F_1|=|F_1\cap F_2|+|F_1\setminus F_2|\ge 3n-7+1=3n-6>2n-3$ for $n\ge 6$, a contradiction to $|F_1|\le 2n-3$.

Case 3.2.3. $u_4^0\ne 00$ and $v_4^0\ne 00$.

By Lemma 2.4, $|N(u)\cap N(v)|\le 4$. Since w_1 is one common neighbor of u and v, $N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)\le 3$. By Lemma 2.3, $|N(u)\cap N(w_1)|=0$ and $|N(v)\cap N(w_1)|=0$. Then $|F_1\cap F_2|\ge|N(u)\setminus\{w_1\}|+|N(v)\setminus\{w_1\}|+|N(w_1)\setminus\{u,v\}|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap|N_{SQ_n[F_1\cap F_2]}(u)|-|N_{SQ_n[F_1\cap F_2]}(w_1)\cap|N_{SQ_n[F_1\cap F_2]}(v)|-|N_{SQ_n[F_1\cap F_2]}(u)\cap N_{SQ_n[F_1\cap F_2]}(v)|\ge n-1+n-1+n-2-0-0-3=3n-7$. Thus, $|F_1|=|F_1\cap F_2|+|F_1\setminus F_2|\ge 3n-7+1=3n-6>2n-3$ for $n\ge 6$, a contradiction to $|F_1|\le 2n-3$.

Therefore, $SQ_n-F_1-F_2$ has no isolated vertex. The proof of Claim 1 is complete.

Let $w\in V(SQ_n)\setminus(F_1\cup F_2)$. By Claim 1, w has at least one neighbor in $SQ_n-F_1-F_2$. Note that the vertex set pair (F_1,F_2) does not satisfy with any one condition of Theorem 3.2. By the condition (1) of Theorem 3.2, for any pair of adjacent vertices $w,u\in V(SQ_n)\setminus(F_1\cup F_2)$, there is no vertex $v\in F_1\Delta F_2$ such that $wv\in E(SQ_n)$. It follows that u has no neighbor in $F_1\Delta F_2$. By the arbitrariness of w, there are no edges between $V(SQ_n)\setminus(F_1\cup F_2)$ and $F_1\Delta F_2$. If $F_1\cap F_2=\varnothing$, then $F_1\Delta F_2=F_1\cup F_2$. This means SQ_n is not connected, a contradiction. Therefore, $F_1\cap F_2\ne\varnothing$ and $F_1\cap F_2$ is a vertex cut of SQ_n. Since F_1 is a 1-good-neighbor faulty set and $F_2\setminus F_1\ne\varnothing$, we have that $\delta(SQ_n[SQ_n-F_1-F_2])\ge 1$ and $\delta(SQ_n[F_2\setminus F_1])\ge 1$. Suppose that $F_1\setminus F_2=\varnothing$. Then $F_1\cap F_2=F_1$. Since F_1 is a 1-good-neighbor faulty set of SQ_n, we have that $F_1\cap F_2=F_1$ is a 1-good-neighbor faulty set of SQ_n. Since there is no edge between $V(SQ_n)\setminus(F_1\cup F_2)$ and $F_2\setminus F_1$, we can deduce that $F_1\cap F_2=F_1$ is a 1-good-neighbor cut of SQ_n. Suppose that $F_1\setminus F_2\ne\varnothing$. Note that $SQ_n-(F_1\cap F_2)$ has three parts (for convenience): H, $SQ_n[F_1\setminus F_2]$ and $SQ_n[F_2\setminus F_1]$. Since F_1,F_2 are two 1-good-neighbor faulty sets of SQ_n, $\delta(H)\ge 1$, $\delta(SQ_n[F_1\setminus F_2])\ge 1$ and $\delta(SQ_n[F_2\setminus F_1])\ge 1$. Obviously, $|F_2\setminus F_1|\ge 2$. Thus, $F_1\cap F_2$ is a 1-good-neighbor cut of SQ_n. By Theorem 2.2, $|F_1\cap F_2|\ge 2n-4$. Then $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\ge 2+2n-4=2n-2$, which contradicts $|F_2|\le 2n-3$. Therefore, SQ_n is 1-good-neighbor $(2n-3)$-diagnosable, i.e., $t_1(SQ_n)\ge 2n-3$.

The proof is complete.

Combining Lemma 3.2 and Lemma 3.4, we have the following theorem.

Theorem 3.4. For $n\ge 6$, the 1-good-neighbor diagnosability of SQ_n under the MM* model is $2n-3$, i.e.,

$t_1(SQ_n) = 2n-3$.

IV. Conclusions

In this paper, we investigate the problem of the 1-good-neighbor diagnosability of the n-dimensional shuffle-cube SQ_n. We determine that the 1-good-neighbor diagnosability of SQ_n under the PMC model and MM * model is $2n-3$ for $n \geq 6$. This work will not only help researchers to discuss g-good-neighbor diagnosability of SQ_n for $g \geq 2$, but also help engineers to develop more different measures of the 1-good-neighbor diagnosability based on application environment and statistics related to faulty patterns.

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